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# Foundation of the Rewriting in an Algebra [1]

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**Abstract.** This paper includes two main ideas. The first one, rewriting in an algebra, was introduced in [5]. The second one, boolean rewriting, can be found in many papers but we were never able to find a clear comparison with the classic one. We prefer rewriting in an algebra to term rewriting. This is our way to give a unique theory of rewriting. If the algebra is free, then we get the term rewriting. If the algebra is a certain quotient of a free algebra then we get rewriting modulo equations. Rewriting is said to be boolean when the condition of each conditional equation is of boolean sort(in the free algebra it is a boolean term). We prove the classic rewriting is equivalent to boolean rewriting in a specific algebra, therefore, boolean rewriting is more general than the classic one.

## 1. Preliminaries

Let $\Sigma$ be an algebraic $S$–sorted signature. Let $X$ be an $S$–sorted set of variables. Let $T_\Sigma(X)$ be the $\Sigma$–algebra freely generated by $X$.

Let $\mathcal{A} = (\{A_s\}_{s\in S}, \{A_\sigma\}_{\sigma\in\Sigma})$ be a $\Sigma$–algebra and let $z$ be a variable of sort $s$, $z \notin A_s$. Let $\mathcal{A}[z]$ be a shorter notation for $T_\Sigma(A \cup \{z\})$ the $\Sigma$–algebra freely generated by $A \cup \{z\}$. An element $c$ from $\mathcal{A}[z]$ is said to be *context* if the number of the occurences of $z$ in $c$ is 1. If $c = \sigma(c_1, c_2, \ldots, c_n)$ is a context then there exists $1 \le i \le n$ such that $c_i$ is a context and $c_j \in T_\Sigma(A)$ for each $j \neq i$.

For $d \in A_s$, let $z \leftarrow d : \mathcal{A}[z] \longrightarrow \mathcal{A}$ be the unique morphism of $\Sigma$–algebras such that $(z \leftarrow d)(z) = d$ and $(z \leftarrow d)(a) = a$ for each $a \in A$. For each $t$ in $\mathcal{A}[z]$ and $a \in A_s$, we prefer to write $t[a]$ instead of $(z \leftarrow a)(t)$.

[1]Partially supported by Softwin

### 1.1. Deduction Rules

We list some families of deduction rules in the set of the formal equalities in a fixed $\Sigma$-algebra $\mathcal{A}$. There is no essential difference between a set of formal equalities in A and a relation in A. These rules are called **R**eflexivity, **T**ransitivity, **C**ompatibility with the operations in $\Sigma$ and **C**ompatibility with each **A**rgument of the operations in $\Sigma$:

**R** $a \doteq_s a$ for each $s \in S$ and $a \in A_s$,

**T** $a \doteq_s b$ and $b \doteq_s c$ imply $a \doteq_s c$ for each $s \in S$ and $a, b, c \in A_s$,

**C**$\Sigma$ $a_i \doteq_{s_i} c_i$ for all $i \in [n]$ imply $A_\sigma(a_1, a_2, \ldots, a_n) \doteq_s A_\sigma(c_1, c_2, \ldots, c_n)$ for each $\sigma \in \Sigma_{s_1 s_2 \ldots s_n, s}$

**CA**$\Sigma$ $a \doteq_{s_i} d$ implies $A_\sigma(a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_n) \doteq_s A_\sigma(a_1, \ldots, a_{i-1}, d, a_{i+1}, \ldots, a_n)$ for each $\sigma \in \Sigma_{s_1 \ldots s_n, s}$, each $i \in [n]$, where $a_j \in A_{s_j}$ for each $j \in \{1, \ldots, i-1, i+1, \ldots, n\}$ and $a, d \in A_{s_i}$.

**Definition 1.** A set of the formal egalities in the algebra $\mathcal{A}$ is said to be reflexive, transitive, compatible with operations or compatible with arguments if it is closed under **R**,**T**, **C**$\Sigma$ or **CA**$\Sigma$, respectively. □

### 1.2. Context Closure

**Definition 2.** A set of the formal egalities $Q$ in the algebra $\mathcal{A}$ is said to be **context closed** if for each context $c$ and for each pair of elements $a, d$ in $A$, $a \doteq_s d \in Q$ implies $c[a] \doteq c[d] \in Q$. □

**Proposition 3.** A relation is **context closed** if and only if it is compatible with arguments.

*Proof.* Suppose $Q$ is context closed. To prove the compatibility with arguments we apply the hypothesis for the context $\sigma(a_1, \ldots, a_{i-1}, z, a_{i+1}, \ldots, a_n)$.

Conversely, the result is proved by structural induction in $\mathcal{A}[z]$ for contexts.

For $c = z$. For each $a \doteq_s d \in Q$, $c[a] = a$, $c[d] = d$, therefore $c[a] \doteq_s c[d] \in Q$.

For a context $c = \sigma(a_1, \ldots, a_{i-1}, c', a_{i+1}, \ldots, a_n)$ where $c' \in \mathcal{A}[z]$ is a context and $a_i \in T_\Sigma(A)$ for each i

$$c[a] = (z \leftarrow a)(c) = A_\sigma(a_1[a], \ldots, a_{i-1}[a], c'[a], a_{i+1}[a], \ldots, a_n[a])$$

and $c[d] = A_\sigma(a_1[d], \ldots, a_{i-1}[d], c'[d], a_{i+1}[d], \ldots, a_n[d])$. Moreover $a_i[a] = a_i[d]$ for each $i$.

From inductive hypothesis $c'[a] \doteq c'[d] \in Q$ and using the compatibility with arguments we get $c[a] \doteq c[d] \in Q$. □

**Definition 4.** For each relation $Q$ on $A$ we denote $\longrightarrow_Q = \{c[a] \doteq c[d] : a \doteq_s d \in Q_s, c \in \mathcal{A}[z]$ is a context where the variable $z$ has the sort $s\}$. Traditionally, we would rather write $a \longrightarrow_Q d$ than $a \doteq d \in \longrightarrow_Q$ . □

**Proposition 5.** $\longrightarrow_Q$ is the least set of formal equalities in $\mathcal{A}$ which is context closed and includes $Q$.

*Proof.* To prove $\longrightarrow_Q$ is context closed, we prefer to show it is compatible with arguments.

Let $c[a] \longrightarrow_Q c[d]$ where $a \doteq d \in Q$, let $\sigma$ be an operation symbol and let $a_i$ be some element in $\mathcal{A}$. Using the context $c' = \sigma(a_1, \dots, a_{i-1}, c, a_{i+1}, \dots, a_n)$ we deduce that $c'[a] \longrightarrow_Q c'[d]$. But as above $c'[a] = A_\sigma(a_1, \dots, a_{i-1}, c[a], a_{i+1}, \dots, a_n)$ and $c'[d] = A_\sigma(a_1, \dots, a_{i-1}, c[d], a_{i+1}, \dots, a_n)$ therefore

$$A_\sigma(a_1, \dots, a_{i-1}, c[a], a_{i+1}, \dots, a_n) \longrightarrow_Q A_\sigma(a_1, \dots, a_{i-1}, c[d], a_{i+1}, \dots, a_n).$$

$Q \subseteq\longrightarrow_Q$ is proved using the context $z$.

If $R$ is context closed, then $Q \subseteq R$ implies $\longrightarrow_Q \subseteq R$. □

### 1.3. Closure under preorder compatible with operations

If a set of formal equalities in A is closed under **C**$\Sigma$ and **R** then it is closed under **CA**$\Sigma$.

**Lemma 6. (Compatibility)** Let $\rho$ be a preorder in A, i.e. it is reflexive and transitive. If $\rho$ is compatible with arguments then $\rho$ is compatible with the operations.

*Proof.* Let $\sigma \in \Sigma_{s_1 \dots s_n, s}$ and $a_i, b_i \in A_{s_i}$ be such that $a_i \doteq_{s_i} b_i \in \rho_{s_i}$ for each $i \in [n]$.

We show that $A_\sigma(a_1, \dots, a_n) \doteq_s A_\sigma(b_1, \dots, b_n) \in \rho_s$.

If $n = 0$ we get $A_\sigma \doteq_s A_\sigma \in \rho_s$ from reflexivity.

If $n = 1$ we get from **CA**$\Sigma$ that $A_\sigma(a_1) \doteq_s A_\sigma(b_1) \in \rho_s$.

If $n \geq 2$ as $\rho$ is compatible with arguments we get for each $1 \leq i \leq n$

$$A_\sigma(b_1, \dots, b_{i-1}, a_i, a_{i+1} \dots, a_n) \doteq_s A_\sigma(b_1, \dots, b_{i-1}, b_i, a_{i+1} \dots, a_n) \in \rho_s$$

From the transitivity of $\rho$ we deduce $A_\sigma(a_1, a_2, \dots, a_n) \doteq_s A_\sigma(b_1, b_2, \dots, b_n) \in \rho_s$. □

We conclude for a reflexive and transitive set of formal equations in $\mathcal{A}$ that it is context closed if and only if it is closed under **CA**$\Sigma$ if and only if it is closed under **C**$\Sigma$.

**Proposition 7.** $\stackrel{*}{\longrightarrow}_Q$ is the least preorder which is compatible with the operations and which includes $Q$.

*Proof.* As $\stackrel{*}{\longrightarrow}_Q$ is reflexive and transitive it suffice to prove the compatibility with arguments, i.e. to show for each $\sigma \in \Sigma_{s_1 \dots s_n, s'}$ that $a \stackrel{*}{\longrightarrow}_Q d$ implies

$$A_\sigma(a_1, \dots, a_{i-1}, a, a_{i+1}, \dots, a_n) \stackrel{*}{\longrightarrow}_Q A_\sigma(a_1, \dots, a_{i-1}, d, a_{i+1}, \dots, a_n).$$

We make an induction on the step number in $a \stackrel{*}{\longrightarrow}_Q d$. We suppose $a \longrightarrow_Q u$ and $u \stackrel{*}{\longrightarrow}_Q d$ with a less step number, therefore the inductive hypothesis implies

$$A_\sigma(a_1, \dots, a_{i-1}, u, a_{i+1}, \dots, a_n) \stackrel{*}{\longrightarrow}_Q A_\sigma(a_1, \dots, a_{i-1}, d, a_{i+1}, \dots, a_n).$$

As $\longrightarrow_Q$ is compatible for arguments, from $a \longrightarrow_Q u$ we deduce

$$A_\sigma(a_1,\ldots,a_{i-1},a,a_{i+1},\ldots,a_n) \longrightarrow_Q A_\sigma(a_1,\ldots,a_{i-1},u,a_{i+1},\ldots,a_n).$$

By transitivity we get

$$A_\sigma(a_1,\ldots,a_{i-1},a,a_{i+1},\ldots,a_n) \stackrel{*}{\longrightarrow}_Q A_\sigma(a_1,\ldots,a_{i-1},d,a_{i+1},\ldots,a_n).$$

Obviously $Q \subseteq \longrightarrow_Q \subseteq \stackrel{*}{\longrightarrow}_Q$ .

Let $R$ be a preorder which is compatible with operations and which includes $Q$. The compatibility with operations implies $\longrightarrow_Q \subseteq R$, therefore $\stackrel{*}{\longrightarrow}_Q \subseteq R$ as $R$ is reflexive and transitive. $\square$

Using the proprieties of the closure operators we deduce that $R \subseteq Q$ implies $\longrightarrow_R \subseteq \longrightarrow_Q$ and $\stackrel{*}{\longrightarrow}_R \subseteq \stackrel{*}{\longrightarrow}_Q$.

Denote $u \downarrow_Q v$ if there exists $a \in \mathcal{A}$ such that $u \stackrel{*}{\longrightarrow}_Q a$ and $v \stackrel{*}{\longrightarrow}_Q a$.

## 1. Classic Style

**Definition 8.** A *conditional equation* is

$$(\forall X) l \doteq_s r \textbf{ if } H$$

where $X$ is a set of S-sorted variables, $l$ and $r$ are two elements of sort $s$ in $T_\Sigma(X)$ and $H$ is a finite set of formal equalities from $T_\Sigma(X)$. $\square$

A conditional equation in which $H = \emptyset$ becomes an unconditional equation and it is said to be an *equation*. In this case we write only $(\forall X) l \doteq_s r$ instead of $(\forall X) l \doteq_s r$ **if** $\emptyset$.

### 2.1. Γ–rewriting

In this section we fix a set $\Gamma$ of conditional equations, called axiomes and a $\Sigma$-algebra $\mathcal{A}$.

We shall use the following deductive rules:

**Rew**$_\Gamma$ For each $(\forall X)\, l \doteq_s r$ **if** $H \in \Gamma$ and for each morphism $h\colon T_\Sigma(X) \to \mathcal{A}$ $(\forall u \doteq_{s'} v \in H)(\exists d \in A_{s'}) h_{s'}(u) \doteq_{s'} d$ and $h_{s'}(v) \doteq_{s'} d$ implies $h_s(l) \doteq_s h_s(r)$.

**SRew**$_\Gamma$ For each $(\forall X)\, l \doteq_s r$ **if** $H \in \Gamma$ and for each morphism $h\colon T_\Sigma(X) \to \mathcal{A}$ $(\forall u \doteq_{s'} v \in H)(\exists d \in A_{s'}) h_{s'}(u) \doteq_{s'} d$ and $h_{s'}(v) \doteq_{s'} d$ implies $c[h_s(l)] \doteq_{s'} c[h_s(r)]$ for each context $c \in \mathcal{A}[z]_{s'}$.

Note that **SRew**$_\Gamma$, *rewriting in a subterm*, is a stronger deductive rule than **Rew**$_\Gamma$, *rewriting*, which is obtained from **SRew**$_\Gamma$ for $c = z$.

If a set of formal equations is closed under $\mathbf{Rew}_\Gamma$ and $\mathbf{CA}\Sigma$ then it is closed under $\mathbf{SRew}_\Gamma$.

We define by induction the increasing sequence of sets of formal equations in $\mathcal{A}$.

$Q_0 = \emptyset$,

$Q_{n+1} = \{h_s(l) \doteq_s h_s(r) \; : \; (\forall Y) l \doteq_s r \textbf{ if } H \in \Gamma, \; h : T_\Sigma(Y) \to \mathcal{A}, \text{ and } (\forall u \doteq v \in H) h(u) \downarrow_{Q_n} h(v)\}$.

By definition $Q$ is the union of the increasing sequence $Q_n$. Note that the sequences $\longrightarrow_{Q_n}$ and $\overset{*}{\longrightarrow}_{Q_n}$ are increasing.

Remark that $a \longrightarrow_Q d$ implies $\exists n$ such that $a \longrightarrow_{Q_n} d$. Also $a \overset{*}{\longrightarrow}_Q d$ implies $\exists n$ such that $a \overset{*}{\longrightarrow}_{Q_n} d$.

**Proposition 9.** $\overset{*}{\longrightarrow}_Q$ is closed under $\mathbf{SRew}_\Gamma$.

*Proof.* Let $(\forall Y) l \doteq_s r \textbf{ if } H$ be in $\Gamma$, let $h : T_\Sigma(Y) \to \mathcal{A}$ be a morphism such that $h_s(u) \downarrow_Q h_s(v)$ for each $u \doteq v \in H$ and a context $c \in \mathcal{A}[z]_{s'}$. We have to show $c[h(l)] \overset{*}{\longrightarrow}_Q c[h(r)]$.

As $H$ is finite and the number of steps used in $h_s(u) \downarrow_Q h_s(v)$ where $u \doteq v \in H$ is finite there exists an $n$ such that $h_s(u) \downarrow_{Q_n} h_s(v)$ for each $u \doteq v \in H$. Therefore $h(l) \doteq h(r) \in Q_{n+1}$. As $h(l) \doteq h(r) \in Q$ we deduce $c[h(l)] \overset{*}{\longrightarrow}_Q c[h(r)]$. □

**Proposition 10.** $\overset{*}{\longrightarrow}_Q$ is the least relation closed under $\mathbf{R}$, $\mathbf{T}$, $\mathbf{C}\Sigma$ and $\mathbf{Rew}_\Gamma$.

*Proof.* Obviously $\overset{*}{\longrightarrow}_Q$ is closed under $\mathbf{R}$, $\mathbf{T}$ and $\mathbf{C}\Sigma$ and it is closed under $\mathbf{Rew}_\Gamma$ because it is closed under $\mathbf{SRew}_\Gamma$.

Let $W$ be a relation closed under $\mathbf{R}$, $\mathbf{T}$, $\mathbf{C}\Sigma$ and $\mathbf{Rew}_\Gamma$. We prove by induction on $n$ that $Q_n \subseteq W$.
If $n = 0$ we have $Q_0 = \emptyset \subseteq W$.
For $n \geq 1$ let $h(l) \doteq h(r) \in Q_n$, where $(\forall Y) l \doteq_s r \textbf{ if } H \in \Gamma, \; h : T_\Sigma(Y) \to \mathcal{A}$ and $(\forall u \doteq v \in H) h_s(u) \downarrow_{Q_{n-1}} h_s(v)$.
By inductive hypothesis $Q_{n-1} \subseteq W$. As $W$ is closed under $C\Sigma$ we deduce that $W$ is context closed, therefore $\longrightarrow_{Q_{n-1}} \subseteq W$. As $W$ is closed under $\mathbf{R}$ and $\mathbf{T}$ we deduce $\overset{*}{\longrightarrow}_{Q_{n-1}} \subseteq W$.
From $(\forall u \doteq v \in H) h_s(u) \downarrow_{Q_{n-1}} h_s(v)$ we get $(\forall u \doteq v \in H) h_s(u) \downarrow_W h_s(v)$. As $W$ is closed under $\mathbf{Rew}_\Gamma$ we deduce $h(l) \doteq h(r) \in W$, therefore $Q_n \subseteq W$.

Hence $Q \subseteq W$ and using proposition 7 we deduce $\overset{*}{\longrightarrow}_Q \subseteq W$. □

In the sequel, when $Q$ is defined as above, traditionally we rather write $\overset{*}{\Longrightarrow}_\Gamma$ than $\overset{*}{\longrightarrow}_Q$. Because of its properties the relation $\overset{*}{\Longrightarrow}_\Gamma$ is said to be $\Gamma$–rewriting or shortly rewriting.

## 3. Boolean Style

We work with a multi-sorted signature $(S, \Sigma)$ with a distinguished sort $b \in S$ ($b$ from *boolean*) and a distinguished constant operation symbol $t \in \Sigma_{\lambda,b}$ ($t$ from *true*).

**Definition 11.** A *conditional equation* is

$$(\forall X) l \doteq_s r \textbf{ if } H$$

where $l$ and $r$ are two elements of sort $s$ and $H$ is an element of sort $b$ from $T_\Sigma(X)$. □

A conditional equation in which $H = T_\Sigma(X)_t$ (the truth of $T_\Sigma(X)$) becomes an unconditional one and it is said to be *equation.* In this case we write only $(\forall X) l \doteq_s r$ instead of $(\forall) l \doteq_s r$ **if** $T_\Sigma(X)_t$.

### 3.1. Boolean Γ–rewriting

In this section we fix a set $\Gamma$ of conditional equations, called axiomes and a $\Sigma$-algebra $\mathcal{A}$.

We shall use the following deductive rules:

**Sub**$_\Gamma$ For each $(\forall X)\, l \doteq_s r$ **if** $H \in \Gamma$ and each morphism $h\colon T_\Sigma(X) \to \mathcal{A}$, $h_b(H) \doteq_b A_t$ implies $h_s(l) \doteq_s h_s(r)$.

**SSub**$_\Gamma$ For each $(\forall X)\, l \doteq_s r$ **if** $H \in \Gamma$ and each morphism $h\colon T_\Sigma(X) \to \mathcal{A}$ $h_b(H) \doteq_b A_t$ implies $c[h_s(l)] \doteq_{s'} c[h_s(r)]$ for each context $c \in \mathcal{A}[z]_{s'}$.

Remark that **SSub**$_\Gamma$(*substitution in a subterm*) is a stronger deductive rule than **Sub**$_\Gamma$(*substitution*) which may be got from **SSub**$_\Gamma$ for $c = z$.

If a set of formal equations are closed under **Sub**$_\Gamma$ and **CA**$\Sigma$ then it is closed under **SSub**$_\Gamma$.

We define by induction the increasing sequence of sets of formal equalities in $\mathcal{A}$.

$B_0 = \emptyset$,

$B_{n+1} = \{h_s(l) \doteq_s h_s(r) : (\forall Y) l \doteq_s r \textbf{ if } H \in \Gamma, h : T_\Sigma(Y) \to \mathcal{A}$ and $h_b(H) \xrightarrow{*}_{B_n} A_t\}$.

By definition $B$ is the union of the increasing sequence $B_n$. Note that the sequences $\longrightarrow_{B_n}$ and $\xrightarrow{*}_{B_n}$ are increasing.

Remark that $a \longrightarrow_B d$ implies $\exists n$ such that $a \longrightarrow_{B_n} d$. Also $a \xrightarrow{*}_B d$ implies $\exists n$ such that $a \xrightarrow{*}_{B_n} d$.

**Fact 12.** $\xrightarrow{*}_B$ is closed under **SSub**$_\Gamma$.

*Proof.* Let $(\forall Y) l \doteq_s r$ **if** $H$ be in $\Gamma$, $h : T_\Sigma(Y) \to \mathcal{A}$ a morphism such that $h_b(H) \xrightarrow{*}_B A_t$ and a context $c \in \mathcal{A}[z]_{s'}$. We have to show that $c[h_s(l)] \xrightarrow{*}_B c[h_s(r)]$.

As the number of steps in $h_b(H) \xrightarrow{*}_B A_t$ is finite there exists an $n$ such that $h_b(H) \xrightarrow{*}_{B_n} A_t$. Therefore $h_s(l) \doteq_s h_s(r) \in B_{n+1}$. As $h_s(l) \doteq_s h_s(r) \in B \subseteq \xrightarrow{*}_B$ and $\xrightarrow{*}_B$ is context closed we deduce $c[h_s(l)] \xrightarrow{*}_B c[h_s(r)]$. □

**Proposition 13.** $\xrightarrow{*}_B$ is the least relation closed under **R**eflexivity, **T**ransitivity, **C**ompatibility with the operations in $\Sigma$ and $\mathbf{Sub}_\Gamma$.

*Proof.* From proposition 7 and fact 12 we deduce that $\xrightarrow{*}_B$ has the above properties.

Let $W$ a relation which is closed under **R**eflexivity, **T**ransitivity, **C**ompatibility with the operation in $\Sigma$ and $\mathbf{Sub}_\Gamma$.

By induction we prove $(\forall n)B_n \subseteq W$. Assume $B_n \subseteq W$. As $W$ is closed under **R, T** and $\mathbf{C}\Sigma$ we deduce $\xrightarrow{*}_{B_n} \subseteq W$. Because $W$ is closed under $\mathbf{Sub}_\Gamma$ we deduce $B_{n+1} \subseteq W$.

Therefore $B \subseteq W$. As $W$ is closed under **R**eflexivity, **T**ransitivity and **C**ompatibility with the operations in $\Sigma$ we deduce $\xrightarrow{*}_B \subseteq W$. □

In the sequel, when $B$ is defined as above, instead of $\xrightarrow{*}_B$ we prefer writing $\overset{*}{\Longrightarrow}_\Gamma$, relation which because of its properties is said to be, in the sequel, boolean $\Gamma$–rewriting or shortly, boolean rewriting.

## 4. Some algebra

We work with a multi-sorted signature $(S, \Sigma)$ which we will enrich with a new sort $b$ (from *boolean*), a constant operation symbol $t :\longrightarrow b$ ($t$ from *true*), an operation symbol $\wedge : bb \longrightarrow b$ and for each $s \in S$ an operation symbol

$$==_s : ss \longrightarrow b.$$

The new signature will be denoted $(S^b, \Sigma^b)$. Obviously $S^b = S \cup \{b\}$.

We want that the operation $\wedge$ be associative, commutative, idempotent having $t$ as neutral element. Let $E^b$ be the set of the equation

$$x \wedge (y \wedge z) = (x \wedge y) \wedge z$$

$$x \wedge y = y \wedge x$$

$$x \wedge x = x$$

$$x \wedge t = x$$

$$t \wedge x = x$$

$$u ==_s u = t$$

where the variables $x, y, z$ have sort $b$ and the variable $u$ has sort $s$.

Remark the inclusion of $\Sigma$ in $(\Sigma^b, E^b)$.

The forgetful functor

$$U : Alg(\Sigma^b, E^b) \longrightarrow Alg(\Sigma)$$

has a left adjoin $\_^b$. Its definition is given in the sequel.

Each $\Sigma$-algebra $\mathcal{A}$ is enriched as a $(\Sigma^b, E^b)$-algebra $\mathcal{A}^b$ defined by:

1. $(A_b, A_\wedge, A_t)$ is the free idempotent commutative monoid generated by $\{a \doteq_s d \ : \ s \in S, a, d \in A_s, a \neq d\}$,
2. operation $==_s$ maps unequal elements $a, d$ in $A_s$ to $a \doteq_s d$, but $a ==_s a = A_t$.

Note that $U(\mathcal{A}^b) = \mathcal{A}$.

**Proposition 14.** For each $(\Sigma^b, E^b)$-algebra $\mathcal{B}$ and for each $\Sigma$-algebra morphism $h : \mathcal{A} \longrightarrow U(\mathcal{B})$ there existe a unique $\Sigma^b$-algebra morphism $h^\# : \mathcal{A}^b \longrightarrow \mathcal{B}$ such that $U(h^\#) = h$.

*Proof.* By definition $h^\#_s = h_s$ for each sort $s$. The function $h^\#_b$ is defined for generators, i.e. for unequal elements $a, d$ in $A_s$, by

$$h^\#_b(a \doteq_s d) = B_{==_s}(h_s(a), h_s(d)),$$

being then extended in a natural way to a monoid morphism by

$$h^\#_b(A_t) = B_t$$

$$h^\#_b(e_1 \ A_\wedge \ e_2 \ A_\wedge \ \ldots \ A_\wedge \ e_n) = h^\#_b(e_1) \ B_\wedge \ h^\#_b(e_2) \ B_\wedge \ldots B_\wedge \ h^\#_b(e_n). \ \Box$$

For each $\Sigma$-algebra morphism $h : \mathcal{A} \longrightarrow \mathcal{B}$ we denote $h^b : \mathcal{A}^b \longrightarrow \mathcal{B}^b$ the $(\Sigma^b, E^b)$-algebra morphism got as above. Remark that $U(h^b) = h$. Therefore $\_^b; U = 1_{Alg_\Sigma}$.

Let $X$ an $S$-sorted set of variables. Assuming that the set of variables of sort $b$ is empty, $X$ may be seen as an $S^b$-sorted of variables denoted in the sequel $X^b$. That is

$$\text{if } s \in S \text{ then } X^b_s = X_s \text{ else } X^b_b = \emptyset.$$

If $B$ is a $S^b$-sorted set then we may identify the sets

$$Set_{S^b}(X^b, B) \quad \text{and} \quad Set_S(X, \{B_s\}_{s \in S}).$$

as for every $S^b$-sorted function defined on $X^b$ its component of sort $b$ must be the inclusion defined on $\emptyset$.

**Proposition 15.** Ler $F : Alg(\Sigma^b) \longrightarrow Alg(\Sigma)$ be the forgetful functor. Let $\mathcal{B}$ be an $\Sigma^b$-algebra and let $X$ be an $S$-sorted set of variables. There exists a bijection between

$$Alg_{\Sigma^b}(T_{\Sigma^b}(X^b), \mathcal{B}) \quad \text{and} \quad Alg_\Sigma(T_\Sigma(X), F(\mathcal{B}))$$

which forgets the component of sort $b$ of the morphism.

*Proof.* The set $Alg_{\Sigma^b}(T_{\Sigma^b}(X^b), \mathcal{B})$ is in a natural bijection with the set $Set_{S^b}(X^b, B)$ and the set $Alg_\Sigma(T_\Sigma(X), F(\mathcal{B}))$ is in a natural bijection with $Set_S(X, \{B_s\}_{s\in S})$). We have seen above that the two sets are as identic. □

If we forget the elements of sort $b$ and the operations those rank involves this sort of a $\Sigma^b$-algebra freely generated by $X^b$ we get an $\Sigma$-algebra freely generated by $X$, i.e. $F(T_{\Sigma^b}(X^b)) = T_\Sigma(X)$.

## 5. Some semantics

In an $\Sigma^b$-algebra $\mathcal{B}$ let

$$G = \{l_1 \doteq_{s1} r_1, l_2 \doteq_{s2} r_2, \dots, l_n \doteq_{sn} r_n\}$$

be a finite set of formal equalities between elements which have not sort $b$. We define an element of sort $b$

$$G^b = l_1 ==_{s1} r_1\ B_\wedge\ l_2 ==_{s2} r_2\ B_\wedge\ \dots\ B_\wedge\ l_n ==_{sn} r_n.$$

If $\mathcal{B}$ is not a $E^b$-algebra, then $G^b$ may not be unique, which case we make a nondeterministic choice.

To pass from classic style to the boolean one we replace each finite set of formal equalities $G$ from $\mathcal{A}$ with $G^b$ an element of sort $b$ in $\mathcal{A}^b$.

We mention the following specific properties of the algebra $\mathcal{A}^b$

1. $x\ A_\wedge\ y = A_t$ implies $x = y = A_t$, for each $x, y \in A_b$,
2. $a ==_s d = A_t$ implies $a = d$ for each $s \in S$ and each $a, d \in A_s$.

**Lemma 16.** If $G$ is a finite set of formal equalities from $T_\Sigma(X)$ and $h : T_{\Sigma^b}(X^b) \longrightarrow \mathcal{A}^b$ is an $\Sigma^b$-morphism then

$$h_b(G^b) = A_t \quad \text{if and only if} \quad h_s(u) = h_s(v) \text{ for each } u =_s v \in G.$$

*Proof.* We use the above notation and we remark the following equivalent facts.

1. $h_b(G^b) = A_t$,
2. $h_b(l_1 ==_{s1} r_1 \wedge l_2 ==_{s2} r_2 \wedge \dots \wedge l_n ==_{sn} r_n) = A_t$,
3. $h_{s1}(l_1)A_{==_{s1}}h_{s1}(r_1)\ A_\wedge\ \dots\ A_\wedge\ h_{sn}(l_n)A_{==_{sn}}h_{sn}(r_n) = A_t$,
4. $h_{si}(l_i)A_{==_{si}}h_{si}(r_i) = A_t$ for each $1 \le i \le n$,
5. $h_{si}(l_i) = h_{si}(r_i)$ for each $1 \le i \le n$. □

Each classic Σ-conditional equation

$$\gamma = (\forall X) l \doteq_s r \textbf{ if } G$$

may be changed in an $\Sigma^b$-conditional equation with a condition of sort $b$

$$\gamma^b = (\forall X^b) l \doteq_s r \textbf{ if } G^b.$$

Let Γ be a set of classic Σ-conditional equations. We denote $\Gamma^b = \{\gamma^b : \gamma \in \Gamma\}$ a set of the $\Sigma^b$-axiomes.

**Proposition 17.** If $\mathcal{A}$ is an Σ-algebra and $\gamma$ an Σ-conditional equation then

$$\mathcal{A}^b \models \gamma^b \text{ if and only if } \mathcal{A} \models \gamma.$$

*Proof.* Using proposition 15 we identify an Σ-morphism $h$ from $T_\Sigma(X)$ to $\mathcal{A}$ and an $\Sigma^b$-morphism from $T_{\Sigma^b}(X^b)$ to $\mathcal{A}^b$. For each such morphism h, from lemma we get the following equivalent facts:

1. if $h_b(l_1 ==_{s1} r_1 \wedge l_2 ==_{s2} r_2 \wedge \ldots \wedge l_n ==_{sn} r_n) = A_t$ then $h_s(l) = h_s(r)$,
2. if $h_{si}(l_i) = h_{si}(r_i)$ for each $1 \leq i \leq n$ then $h_s(l) = h_s(r)$,

and the conclusion follows easily. □

**Corollary 18.** If $\mathcal{A}$ is an Σ-algebra and Γ a set of Σ-conditional equations then

$$\mathcal{A} \models_\Sigma \Gamma \quad \text{if and only if} \quad \mathcal{A}^b \models_{\Sigma^b} \Gamma^b \ . \ \square$$

## 6. Boolean Versus Classic

We show that the classic Γ-rewritings in the Σ-algebra $\mathcal{A}$ have the same power as the boolean $\Gamma^b$-rewritings in the $\Sigma^b$-algebra $\mathcal{A}^b$.

**Theorem 19.** 1. For each $s \in S$, for each $u, v \in A_s$

$$u \stackrel{*}{\Longrightarrow}_\Gamma v \text{ in } \mathcal{A} \text{ if and only if } u \stackrel{*}{\Longrightarrow}_{\Gamma^b} v \text{ in } \mathcal{A}^b.$$

2. For $G = \{l_1 \doteq_{s1} r_1, l_2 \doteq_{s2} r_2, \ldots, l_n \doteq_{sn} r_n\}$ a set of formal equations with elements of sorts notequal to $b$

$$(\forall i) l_i \downarrow_\Gamma r_i \text{ in } \mathcal{A} \text{ if and only if } G^b \stackrel{*}{\Longrightarrow}_{\Gamma^b} A_t \text{ in } \mathcal{A}^b.$$

*Proof.* 1. Assume $u \stackrel{*}{\Longrightarrow}_\Gamma v$ in $\mathcal{A}$. Using the notation from the section "Classic Style" this means $u \stackrel{*}{\longrightarrow}_Q v$. As $Q$ is the union of the sequence $\{Q_n\}$ there exists a natural number $n$ such that $u \stackrel{*}{\longrightarrow}_{Q_n} v$.

By induction by $n$. If $n = 0$ then $u = v$ therefore $u \stackrel{*}{\Longrightarrow}_{\Gamma^b} v$ in $\mathcal{A}^b$.

Assume $u \xrightarrow{*}_{Q_{n+1}} v$. We do another induction on the number of rewriting steps. In case 0 $u = v$.

Suppose $u \xrightarrow{*}_{Q_{n+1}} w$ and $w \longrightarrow_{Q_{n+1}} v$. Moreover by the inductive hypothesis $u \overset{*}{\Longrightarrow}_{\Gamma^b} w$ în $\mathcal{A}^b$.
From $w \longrightarrow_{Q_{n+1}} v$ we deduce $w = c[a]$ and $v = c[d]$ where $a \doteq d \in Q_{n+1}$. Therefore there exist $(\forall Y) l \doteq_s r$ **if** $H \in \Gamma$ and the morphism $h : T_\Sigma(Y) \to \mathcal{A}$ such that $(\forall p \doteq q \in H) h_s(p) \downarrow_{Q_n} h_s(q)$, $a = h_s(l)$ and $d = h_s(r)$.

Therefore for each $p \doteq q \in H$ there exists $g_{pg}$ such that $h_s(p) \xrightarrow{*}_{Q_n} g_{pg}$ and $h_s(q) \xrightarrow{*}_{Q_n} g_{pg}$. From the inductive hypothesis $h_s(p) \overset{*}{\Longrightarrow}_{\Gamma^b} g_{pg}$ and $h_s(q) \overset{*}{\Longrightarrow}_{\Gamma^b} g_{pg}$. We deduce $h_s(p) ==_s h_s(q) \overset{*}{\Longrightarrow}_{\Gamma^b} g_{pg} ==_s g_{pg} = A_t$.
From proposition 15 morphism $h$ may be seen as a $\Sigma^b$-algebra morphism from $T_{\Sigma^b}(Y^b)$ to $\mathcal{A}^b$. Therefore $h_b(H^b) \overset{*}{\Longrightarrow}_{\Gamma^b} A_t$.

We deduce $c[h_s(l)] \Longrightarrow_{\Gamma^b} c[h_s(r)]$, i.e. $w \Longrightarrow_{\Gamma^b} v$, therefore $u \overset{*}{\Longrightarrow}_{\Gamma^b} v$.

For the converse we assume $u \overset{*}{\Longrightarrow}_{\Gamma^b} v$ in $\mathcal{A}^b$. Using the notation in the section "Boolean Style" this means $u \xrightarrow{*}_B v$. As $B$ is the union of the sequence $\{B_n\}$ there exists a natural number $n$ such that $u \xrightarrow{*}_{B_n} v$.

By induction on $n$. If $n = 0$ then $u = v$ therefore $u \overset{*}{\Longrightarrow}_\Gamma v$ in $\mathcal{A}$.

Assume $u \xrightarrow{*}_{B_{n+1}} v$. We do another induction on the number of rewriting steps. For 0 steps $u = v$.

Suppose $u \xrightarrow{*}_{B_{n+1}} w$ and $w \longrightarrow_{B_{n+1}} v$. Moreover by the inductive hypothesis $u \overset{*}{\Longrightarrow}_\Gamma w$ in $\mathcal{A}$. From $w \longrightarrow_{B_{n+1}} v$ we deduce $w = c[a]$ and $v = c[d]$ where $a \doteq d \in B_{n+1}$. Therefore there exist

$$(\forall Y) l \doteq_s r \textbf{ if } \{l_1 \doteq_{s1} r_1, l_2 \doteq_{s2} r_2, \ldots, l_k \doteq_{sk} r_k\} \in \Gamma$$

and the $\Sigma^b$-morphism $h : T_{\Sigma^b}(Y^b) \to \mathcal{A}^b$, such that

$$h_b(l_1 ==_{s1} r_1 \wedge l_2 ==_{s2} r_2 \wedge \ldots \wedge l_k ==_{sk} r_k) \xrightarrow{*}_{B_n} A_t,$$

$a = h_s(l)$ and $d = h_s(r)$. We deduce

$$h_{s1}(l_1) ==_{s1} h_{s1}(r_1) \wedge h_{s2}(l_2) ==_{s2} h_{s2}(r_2) \wedge \ldots \wedge h_{sk}(l_k) ==_{sk} h_{sk}(r_k) \xrightarrow{*}_{B_n} A_t.$$

As the $\Gamma^b$-rewriting can not be made at top and the algebra $\mathcal{A}^b$ has specific properties $h_{si}(l_i) ==_{si} h_{si}(r_i) \xrightarrow{*}_{B_n} A_t$ for each $1 \leq i \leq k$. As the $\Gamma^b$-rewriting can not be made at top in $h_{si}(l_i) ==_{si} h_{si}(r_i)$ and the algebra $\mathcal{A}^b$ has specific properties for each $1 \leq i \leq k$ there exists $a_i \in A_{si}$ such that

$$h_{si}(l_i) \xrightarrow{*}_{B_n} a_i \text{ and } h_{si}(r_i) \xrightarrow{*}_{B_n} a_i.$$

By the inductive hypothesis $h_{si}(l_i) \overset{*}{\Longrightarrow}_\Gamma a_i$ and $h_{si}(r_i) \overset{*}{\Longrightarrow}_\Gamma a_i$, therefore $h_{si}(l_i) \downarrow_\Gamma h_{si}(r_i)$ for each $1 \leq i \leq k$, then $w \Longrightarrow_\Gamma v$, hence $u \overset{*}{\Longrightarrow}_\Gamma v$.

2. Assume $l_i \downarrow_\Gamma r_i$ în $\mathcal{A}$ for each $i$. There exists $u_i$ such that $l_i \overset{*}{\Longrightarrow}_\Gamma u_i$ and $r_i \overset{*}{\Longrightarrow}_\Gamma u_i$. Using the first item of the theorem we get $l_i \overset{*}{\Longrightarrow}_{\Gamma^b} u_i$ and $r_i \overset{*}{\Longrightarrow}_{\Gamma^b} u_i$. We deduce

$l_1 ==_{s1} r_1 \wedge l_2 ==_{s2} r_2 \wedge \ldots \wedge l_n ==_{sn} r_n \overset{*}{\Longrightarrow}_{\Gamma^b} u_1 ==_{s1} u_1 \wedge \ldots \wedge u_n ==_{sn} u_n = = A_t \wedge \ldots \wedge A_t = A_t$.

For the converse, suppose that

$$l_1 ==_{s1} r_1 \wedge l_2 ==_{s2} r_2 \wedge \ldots \wedge l_n ==_{sn} r_n \overset{*}{\Longrightarrow}_{\Gamma^b} A_t.$$

As the $\Gamma^b$-rewriting can not be made at top and the algebra $\mathcal{A}^b$ has specific properties we deduce for each $1 \leq i \leq n$ that $l_i ==_{si} r_i \overset{*}{\Longrightarrow}_{\Gamma^b} A_t$.

As the $\Gamma^b$-rewriting can not be made at top in $l_i ==_{si} r_i$ and the algebra $\mathcal{A}^b$ has specific properties we deduce that for each $1 \leq i \leq n$ there exists $u_i$ such that $l_i \overset{*}{\Longrightarrow}_{\Gamma^b} u_i$ and $r_i \overset{*}{\Longrightarrow}_{\Gamma^b} u_i$. Using the first item of the theorem we get $l_i \downarrow_\Gamma r_i$. □

**Corollary 20.** For each $s \in S$ and for each $u, v \in A_s$

$$u \Longrightarrow_\Gamma v \text{ în } \mathcal{A} \text{ if and only if } u \Longrightarrow_{\Gamma^b} v \text{ în } \mathcal{A}^b.$$

*Proof.* For each $(\forall Y)l \doteq_s r$ **if** $H \in \Gamma$ and for each $\Sigma^b$-morphism $h : T_{\Sigma^b}(Y^b) \to \mathcal{A}^b$, using the second conclusion of the theorem 19 applied to the set $h(H)$ and the equality $h(H)^b = h_b(H^b)$ we deduce

$$(\forall u = v \in H)h(u) \downarrow_\Gamma h(v) \text{ in } \mathcal{A} \text{ if and only if } h_b(H^b) \overset{*}{\Longrightarrow}_{\Gamma^b} A_t \text{ in } \mathcal{A}^b.$$

The conclusion follows easily applying one step rewriting definitions. □

The above propositions prove that the classic rewriting in an $\Sigma$-algebra $\mathcal{A}$ is obtained by boolean rewriting in the $\Sigma^b$-algebra $\mathcal{A}^b$.

## 17. Conclusion

As the classic rewriting is equivalent to boolean rewriting in a specific algebra we get the conclusion that the boolean rewriting is more general than the classic rewriting.

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