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Completeness of Paramodulation without Lifting Lemma

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Abstract. Equational logic programming deals with systems of equations in many-sorted algebras. Paramodulation is the main rule used for obtaining solutions. The lifting lemma is the most difficult step for proving the completeness. We prove without using a lifting lemma that every solution may be found using the paramodulation only.

1. Introduction

An equation is a formal equality of the form $l\dot{=}r$ in $T_\Sigma(X)$, the absolutely free algebra generated by the many-sorted set of variables X. Let G be a finite set of equations from $T_\Sigma(X)$. The problem of equational logic programming is $(\exists X)G$.

We search the solution in the class of Γ-algebras, i.e. algebras satisfying a set of conditional equations Γ. The morphism $\sigma : T_\Sigma(X) \to T_\Sigma(Y)$ is said to be a *solution* if $\Gamma \models (\forall Y)\sigma(G)$.

The paramodulation is the main method used for finding solutions. We recall this rule. To apply this rule to a set of equations from $T_\Sigma(X)$ we use an axiom $(\forall Y)l\dot{=}r \; if \; H \in \Gamma$ (H is a set of equations from $T_\Sigma(Y)$ and $l\dot{=}r$ is an equation in $T_\Sigma(Y)$), where $Y \cap X \neq \emptyset$. The rule is:

$$G \cup C[a] \longrightarrow_p \theta(G \cup H \cup C[r]),$$

where $\theta : T_\Sigma(X \cup Y) \to T_\Sigma(Z)$ is the most general unifier of a and l, G is a set of equations from $T_\Sigma(X)$ and C is an extended context from $T_\Sigma(X \cup \{z\})$, where z is a new variable which appears only once in C. $C[a]$ and $C[r]$ are the results of the substitution of a and r for z, respectively.

It is known that in the proof of completeness the lifting lemma is the most difficult step. We prove without using a lifting lemma that every solution may be found using the paramodulation only.

1. Preliminaries

Let (S, Σ) be a many-sorted signature. For a Σ-algebra $\mathcal{A}$, let $Sen(\mathcal{A}) = \{a \dot{=}_s b \mid s \in S, a, b \in A_s\}$ be the set of formal equalities in $\mathcal{A}$, i.e. the sentences of $\mathcal{A}$.

In the sequel, we fix Γ a set of conditional equations and $G \subseteq Sen(\mathcal{A})$.

We will denote by $h(G) \subseteq \Delta_M$ the fact that $h_s(a) = h_s(b)$, for any $a \dot{=}_s b \in G$, where $h : \mathcal{A} \to \mathcal{M}$.

As we use algebras for quantifications, it is useful to recall some definitions:

$\mathcal{B} \models (\forall \mathcal{A})G \Leftrightarrow (\forall h : \mathcal{A} \to \mathcal{B})\ h(G) \subseteq \Delta_B$;

$\Gamma \models (\forall \mathcal{A})G \Leftrightarrow (\forall \mathcal{B} \models \Gamma)\ \mathcal{B} \models (\forall \mathcal{A})G$, i.e. for any Σ-morphism $h : \mathcal{A} \to \mathcal{B} \models \Gamma$, $h(G) \subseteq \Delta_B$;

$\mathcal{B} \models (\exists \mathcal{A})G \Leftrightarrow (\exists h : \mathcal{A} \to \mathcal{B})$, $h(G) \subseteq \Delta_B$;

$\Gamma \models (\exists \mathcal{A})G \Leftrightarrow (\forall \mathcal{B} \models \Gamma)\ \mathcal{B} \models (\exists \mathcal{A})G$.

The *semantic congruence* associated with a Σ-algebra $\mathcal{A}$, denoted by $\equiv^{\mathcal{A}}_{\Gamma}$, is defined by:

$$a \equiv^{\mathcal{A}}_{\Gamma} b \Leftrightarrow \Gamma \models a \dot{=}_s b \Leftrightarrow h_s(a) = h_s(b), \text{ for any } h : \mathcal{A} \to \mathcal{B} \models \Gamma.$$

We can easily see that

$$\Gamma \models (\forall \mathcal{A})G \text{ if and only if } G \subseteq \equiv^{\mathcal{A}}_{\Gamma}.$$

Proposition 2.1. *If $h : \mathcal{A} \to \mathcal{B}$ is a Σ-morphism, then $h(\equiv^{\mathcal{A}}_{\Gamma}) \subseteq \equiv^{\mathcal{B}}_{\Gamma}$.*

Proof. Let $a \equiv^{\mathcal{A}}_{\Gamma} b$ and $f : \mathcal{B} \to \mathcal{M} \models \Gamma$. Because $h; f : \mathcal{A} \to \mathcal{M} \models \Gamma$, it follows that $(h; f)(a) = (h; f)(b)$, equivalent to $f(h(a)) = f(h(b))$. Because f is arbitrary, we deduce that $h(a) \equiv^{\mathcal{B}}_{\Gamma} h(b)$. □

Corollary 2.1. *If $\Gamma \models (\forall \mathcal{A})G$ and $h : \mathcal{A} \to \mathcal{B}$ is a Σ-morphism, then $\Gamma \models (\forall \mathcal{B})h(G)$.*

Let $\mathcal{A}$ be a Σ-algebra and z a new variable of sort s such that $z \notin A_s$. We will denote the free Σ-algebra generated by $A \cup \{z\}$, $T_\Sigma(A \cup \{z\})$, by $\mathcal{A}[z]$. An element c of $\mathcal{A}[z]$ is called a <u>*context*</u> if the number of occurrences of z in c is 1. For any $d \in A_s$, we note $(z \leftarrow d) : \mathcal{A}[z] \to \mathcal{A}$ the unique Σ-morphism with the property $(z \leftarrow d)(z) = d$ and $(z \leftarrow d)(a) = a$, for any $a \in A$. For any $t \in \mathcal{A}[z]$ and $a \in A$, we shall write $t[a]$ instead of $(z \leftarrow a)(t)$.

Let $c \in \mathcal{A}[z]_s$ be a context and $v \in A_s$. A formal equation of the form $c \dot{=}_s v$ or $v \dot{=}_s c$ is called an <u>*extended context*</u>. An extended context $c \dot{=}_s v$ (or $v \dot{=}_s c$) will be denoted by C. We can easily observe that $(c \dot{=}_s v)[a] = (z \leftarrow a)(c \dot{=}_s v) = ((z \leftarrow a)(c) \dot{=}_s (z \leftarrow a)(v)) = (c[a] \dot{=}_s v)$.

Any Σ-morphism $h : \mathcal{A} \to \mathcal{B}$ can be uniquely extended to a Σ-morphism $h^z : \mathcal{A}[z] \to \mathcal{B}[z]$ by $h^z(z) = z$ and $h^z(a) = h(a)$, for any $a \in A$. We have $(z \leftarrow$

$a); h = h^z; (z \leftarrow h(a))$, for any $a \in A$. For a context $c \in \mathcal{A}[z]$, it follows that $h(c[a]) = h^z(c)[h(a)]$, where $h^z(c)$ is a context.

If C is an extended context, then $h^z(C)$ is also an extended context and

$$h(C[a]) = h^z(C)[h(a)].$$

If $G \subseteq Sen(\mathcal{A})$, for an arbitrary Σ-algebra $\mathcal{A}$, then a Σ-morphism $u : \mathcal{A} \to \mathcal{B}$ is called a *unifier* of G if $u_s(a) = u_s(b)$, for any $a \doteq_s b \in G$. An Σ-morphism $U : \mathcal{A} \to \mathcal{B}$ is called *the most general unifier* of G if U is an unifier for G and for any other unifier $u : \mathcal{A} \to \mathcal{C}$, there exists a Σ-morphism $\sigma : \mathcal{B} \to \mathcal{C}$ such that $u = U; \sigma$. We shall denote by $U = mgu\{G\}$ the fact that U is the most general unifier of G.

3. Solutions and deduction rules

Definition 3.1. A Σ-*morphism* $s : \mathcal{A} \to \mathcal{B}$ is called a **solution** for $(\exists \mathcal{A})G$ if $\Gamma \models (\forall \mathcal{B})s(G)$.

Proposition 3.1. *The composition of a solution with any Σ-morphism is also a solution.*

Proof. Let $s : \mathcal{A} \to \mathcal{B}$ be a solution for $(\exists \mathcal{A})G$ and $h : \mathcal{B} \to \mathcal{C}$ a Σ-morphism. We must prove that $s; h : \mathcal{A} \to \mathcal{C}$ is a solution for $(\exists \mathcal{A})G$, which is equivalent to show that $(s; h)(G) \subseteq \equiv_\Gamma^{\mathcal{C}}$.

Because $s : \mathcal{A} \to \mathcal{B}$ is a solution for $(\exists \mathcal{A})G$, it follows that $s(G) \subseteq \equiv_\Gamma^{\mathcal{B}}$. By Proposition 2.1, we deduce that $h(\equiv_\Gamma^{\mathcal{B}}) \subseteq \equiv_\Gamma^{\mathcal{C}}$, therefore $h(s(G)) \subseteq \equiv_\Gamma^{\mathcal{C}}$. Hence $s; h$ is a solution for $(\exists \mathcal{A})G$. □

By the last proposition, it follows that if we have a solution, then it is not unique. So, we are interested in finding a solution as general as possible.

Usually, solutions are built step by step, appearing in the end as a composition of several morphisms. The morphisms that appear in the process and which we hope to provide a solution, are called *calculated morphisms*.

In the following we shall present some correct deduction rules used in logic programming. These rules allow us to go from a set G of formal equalities to another set G' of formal equalities, obtaining also a calculated morphism. We shall stop applying these rules when we get to a set of real equalities. At this point, we can compose all the morphisms obtained so far, resulting a solution for the initial problem. This thesis will be proved later by showing that the deduction rules are correct.

The rule of morphism: *If $G \subseteq Sen(T_\Sigma(X))$ and $\theta : T_\Sigma(X) \to T_\Sigma(Y)$, then*

$$G \longrightarrow_m \theta(G),$$

with the calculated morphism θ.

The rule of pararewriting: *Let $G \subseteq Sen(T_\Sigma(X))$, $(\forall Y) l \doteq_s r$* if *$H \in \Gamma$ and $\theta : T_\Sigma(Y) \to T_\Sigma(X)$. If C is an extended context with variable z of sort s, then*

$$G \cup \{C[\theta_s(l)]\} \longrightarrow_{pr} G \cup \theta(H) \cup \{C[\theta_s(r)]\},$$

and the calculated morphism is the identity of $T_\Sigma(X)$*.*

The rule of extended paramodulation: *Let* $(\forall Y) l \doteq_s r$ if $H \in \Gamma$*. Let* X *such that* $X \cap Y = \emptyset$*,* $G \subseteq Sen(T_\Sigma(X))$ *and* $\theta : T_\Sigma(X \cup Y) \to T_\Sigma(Z)$ *a* Σ*-morphism such that* $\theta_s(l) = \theta_s(a)$*, where* $a \in T_\Sigma(X)_s$*. If* C *is an extended context with variable* z *of sort* s*, then*

$$G \cup \{C[a]\} \longrightarrow_{ep} \theta(G \cup H \cup \{C[r]\}),$$

with the calculated morphism $\theta_{/X}$*, the restriction of* θ *at* $T_\Sigma(X)$*.*

The rule of paramodulation: *Let* $(\forall Y) l \doteq_s r$ if $H \in \Gamma$*. Let* X *such that* $X \cap Y = \emptyset$*,* $G \subseteq Sen(T_\Sigma(X))$ *and* $\theta : T_\Sigma(X \cup Y) \to T_\Sigma(Z)$ *a* Σ*-morphism such that* $\theta = mgu\{l, a\}$*, where* $a \in T_\Sigma(X)_s$*. If* C *is an extended context with variable* z *of sort* s*, then*

$$G \cup \{C[a]\} \longrightarrow_{p} \theta(G \cup H \cup \{C[r]\}),$$

with the calculated morphism $\theta_{/X}$*, the restriction of* θ *at* $T_\Sigma(X)$*.*

4. Connections between the deduction rules

In the following we shall establish some connections between the deduction rules. These results can help us prove different properties for the deduction rules.

First of all, it is obvious that the rule of paramodulation is a particular case of that of extended paramodulation.

Proposition 4.1. *The rule of pararewriting is a particular case of the rule of extended paramodulation, where the calculated morphism is the identity.*

Proof. We consider the pararewriting

$$G \cup \{C[h_s(l)]\} \longrightarrow_{pr} G \cup h(H) \cup \{C[h_s(r)]\},$$

where $(\forall Y) l \doteq_s r$ if $H \in \Gamma$ and $h : T_\Sigma(Y) \to T_\Sigma(X)$ is a Σ-morphism (suppose $X \cap Y = \emptyset$).

Let $a = h_s(l)$. We consider the Σ-morphism $\theta : T_\Sigma(X \cup Y) \to T_\Sigma(X)$ defined by:

1. $\theta(y) = h(y)$, for any $y \in Y$,
2. $\theta(x) = x$, for any $x \in X$.

It is obvious that $\theta(t) = t$, for any $t \in T_\Sigma(X)$, and $\theta(u) = h(u)$, for any $u \in T_\Sigma(Y)$.

We can easily observe that $\theta(G) = G$, $\theta(H) = h(H)$, $\theta^z(C) = C$, $\theta_s(r) = h_s(r)$ and $\theta_s(a) = \theta_s(l)$. Therefore we can apply the rule of extended paramodulation for $(\forall Y) l \doteq_s r$ if $H \in \Gamma$ and $\theta : T_\Sigma(X \cup Y) \to T_\Sigma(X)$:

$$G \cup \{C[h_s(l)]\} = G \cup \{C[a]\} \longrightarrow_{ep} \theta(G \cup H \cup \{C[r]\}) = \theta(G) \cup \theta(H) \cup \theta(C[r]) =$$

$$= G \cup h(H) \cup \theta^z(C)[\theta_s(r)] = G \cup h(H) \cup C[h_s(r)].$$

The calculated morphism $\theta_{/X}$ is the identity of $T_\Sigma(X)$. □

Proposition 4.2. *If for any* $(\forall Y)l \doteq_s r$ if H *in* Γ*, any variable from* Y *appears in* l*, then the rule of pararewriting is a particular case of the rule of paramodulation, where the calculated morphism is the identity.*

Proof. In the framework of the proof of the last proposition, it is enough to show that θ is the most general unifier for l and a.

Let $u : T_\Sigma(X \cup Y) \to T_\Sigma(Z)$ be an unifier for l and a. Because $u_s(l) = u_s(a) = u_s(h_s(l))$ and any variable from Y appears in l, we get that $u(y) = u(h(y))$, for any $y \in Y$.

Let us denote by $u_{|X}$ the restriction of u at X. By definition $u_{|X}(x) = u(x)$, for any $x \in X$.

Furthermore, for any $x \in X$, $u_{|X}(\theta(x)) = u_{|X}(x) = u(x)$, and for $y \in Y$, $u_{|X}(\theta(y)) = u_{|X}(h(y)) = u(h(y)) = u(y)$. Thus $\theta; u_{|X} = u$.

Hence θ is the most general unifier for l and a. □

Lemma 4.1. *If* $(\forall Y)t \doteq_s t \in \Gamma$*, then* $G \longrightarrow_p (x \leftarrow t)(G)$*, where* x *is a variable which appears in* G *and does not appear in* t*.*

Proof. We choose an occurrence of x in G. Therefore $G = G' \cup C[x]$, where C is an extended context. By the rule of paramodulation applied for $(\forall Y)t \doteq_s t \in \Gamma$, $a = x$, $\theta = mgu\{a, l\} = mgu\{x, t\} = x \leftarrow t$, we get:

$$G = G' \cup C[x] \longrightarrow_p (x \leftarrow t)(G' \cup C[t]) = (x \leftarrow t)(G).$$

The last equality is true because x does not appear in t. □

Lemma 4.2. (The substitution lemma) *If the following conditions are fulfilled:* G *is a finite set,* $(\forall x)x \doteq x \in \Gamma$*, for any variable* x*,* $(\forall x_1\, \forall x_2 \ldots \forall x_n) f(x_1, x_2, \ldots, x_n) \doteq f(x_1, x_2, \ldots, x_n) \in \Gamma$*, for any operation symbol* f*, then the rule of morphism can be obtained by the rule of paramodulation.*

Proof. We prove the lemma in three steps:

(1) The substitution of any other variable y for a variable x can be obtained by the rule of paramodulation if we have the axiom $(\forall y)y \doteq y$.

If x appears in G and $x \neq y$, then we apply Lemma 4.1.

(2) The substitution of a term t for a variable x can be obtained by the rule of paramodulation in the presence of the axioms $(\forall x)x \doteq x$ and

$$(\forall x_1\, \forall x_2 \ldots \forall x_n) f(x_1, x_2, \ldots, x_n) \doteq f(x_1, x_2, \ldots, x_n).$$

We prove this step by structural induction on the complexity of t.

The basic step is just (1). Let us suppose that $t = f(t_1, t_2, \ldots, t_n)$. If x appears in G, by using $(\forall x_1\, \forall x_2 \ldots \forall x_n) f(x_1, x_2, \ldots, x_n) \doteq f(x_1, x_2, \ldots, x_n) \in \Gamma$, where $x_1, \ldots, x_n$ are new variables, and Lemma 4.1, we infer that

$$G \longrightarrow_p (x \leftarrow f(x_1, x_2, \ldots, x_n))(G).$$

We can apply the induction hypothesis for the substitution of each term t_i for each variable x_i, for any $1 \leq i \leq n$. It is also obvious that

$$x \leftarrow f(x_1, x_2, \ldots, x_n) \; ; \; x_1 \leftarrow t_1 \; ; \; x_2 \leftarrow t_2 \; ; \; \ldots \; ;$$

$$x_n \leftarrow t_n = x \leftarrow f(t_1, t_2, \ldots, t_n),$$

because $x_1, x_2, \ldots, x_n$ are new variables.

(3) The rule of morphism can be obtained by the rule of paramodulation.
Let $h : T_\Sigma(X) \to T_\Sigma(Y)$. Because $var(G) = \{x_1, x_2, \ldots, x_n\} \subseteq X$, the rule of morphism is nothing more than the substitution of $h(x_i)$ for each variable x_i, $1 \leq i \leq n$. We can obtain this by using (1) and (2):

- first, we substitute a new variable z_i for every variable x_i, for any $1 \leq i \leq n$:

$$G \longrightarrow_p (x_1 \leftarrow z_1)(G) \longrightarrow_p \ldots \longrightarrow_p (x_n \leftarrow z_n)(\ldots(x_1 \leftarrow z_1)(G)\ldots) = G'$$

- then we substitute $h(x_i)$ for every variable z_i, for any $1 \leq i \leq n$:

$$G' \overset{*}{\longrightarrow}_p (z_1 \leftarrow h(x_1))(G') \overset{*}{\longrightarrow}_p \ldots$$

$$\overset{*}{\longrightarrow}_p (z_n \leftarrow h(x_n))(\ldots(z_1 \leftarrow h(x_1))(G')\ldots) = h(G). \;\square$$

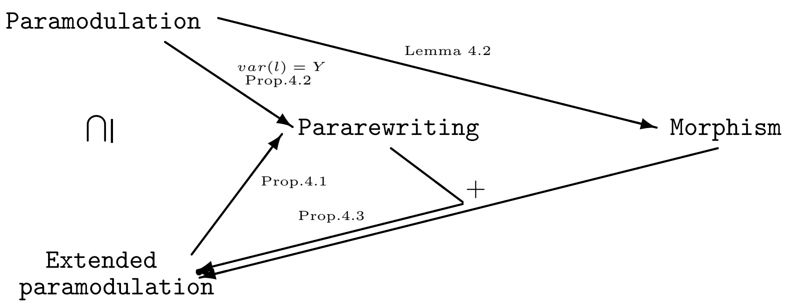

Fig. 1. Connections between the deduction rules.

Proposition 4.3. *The rule of extended paramodulation can be obtained from the rule of morphism and the rule of pararewriting.*

Proof. Let $(\forall Y)l \doteq_s r$ if $H \in \Gamma$, $\theta : T_\Sigma(X \cup Y) \to T_\Sigma(Z)$ such that $\theta_s(l) = \theta_s(a)$, where $a \in T_\Sigma(X)_s$. Let C be an extended context.

We can apply the rule of morphism for θ and we obtain:

$$G \cup \{C[a]\} \longrightarrow_m \theta(G) \cup \{\theta(C[a])\}.$$

It is easy to see that

$$\theta(C[a]) = \theta^z(C)[\theta(a)] = \theta^z(C)[\theta(l)].$$

Now we can apply the rule of pararewriting and we get:

$\theta(G) \cup \{\theta^z(C)[\theta(l)]\} \longrightarrow_{pr} \theta(G) \cup \theta(H) \cup \{\theta^z(C)[\theta(r)]\} = \theta(G \cup H \cup \{C[r]\})$.

So, $G \cup \{C[a]\} \longrightarrow_m \theta(G) \cup \{\theta^z(C)[\theta(l)]\} \longrightarrow_{pr} \theta(G \cup H \cup \{C[r]\})$. □

We can express the connections between the deduction rules through the Fig. 1.

5. Correctness of the deduction rules

Definition 5.1. Let R be a deduction rule. Suppose that if we apply rule R, we get $G \longrightarrow_R G'$ with the calculated morphism θ. We say that R is a *correct rule* if the following condition holds:

if S is a solution for G', then $\theta; S$ is a solution for G.

If we apply only correct rules and we get to a set of true equalities, then the composition of all the calculated morphisms is a solution for the initial problem. This statement is true because the identity morphism is a solution for any set of true equalities, even for the empty set.

In the following we will show that all the deduction rules used in the previous sections are correct.

Proposition 5.1. *The rule of morphism is correct.* □

Proposition 5.2. *The rule of pararewriting is correct.*

Proof. Let us consider the pararewriting

$$G \cup C[\theta_s(l)] \longrightarrow_{pr} G \cup \theta(H) \cup \{C[\theta_s(r)]\},$$

where $(\forall Y) l \doteq_s r$ if $H \in \Gamma$ and $\theta : T_\Sigma(Y) \to T_\Sigma(X)$ is a Σ-morphism.

Let $S : T_\Sigma(X) \to \mathcal{B}$ be a solution for $(\exists X) G \cup \theta(H) \cup \{C[\theta_s(r)]\}$, i.e.

$$S(G \cup \theta(H) \cup \{C[\theta_s(r)]\}) \ \subseteq \ \equiv_\Gamma^\mathcal{B} . \qquad (1)$$

We must show that $S : T_\Sigma(X) \to \mathcal{B}$ is a solution for $(\exists X)(G \cup C[\theta_s(l)])$, i.e.

$$S(G \cup \{C[\theta_s(l)]\}) \ \subseteq \ \equiv_\Gamma^\mathcal{B} .$$

As $S(G) \subseteq \equiv_\Gamma^\mathcal{B}$ follows immediately from (1), it is enough to show that

$$S(C[\theta_s(l)]) \in \equiv_\Gamma^\mathcal{B} .$$

Let $h : \mathcal{B} \to \mathcal{M} \models \Gamma$. From (1) we get that

$$(S; h)(G) \cup (\theta; S; h)(H) \cup \{(S; h)(C[\theta_s(r)])\} \subseteq \Delta_M .$$

Therefore

$$(\theta; S; h)(H) \subseteq \Delta_M , \qquad (2)$$

$$(S;h)(C[\theta_s(r)]) \in \Delta_M. \tag{3}$$

Because $\mathcal{M} \models (\forall Y)l \doteq_s r$ if H, $\theta; S; h : T_\Sigma(Y) \to \mathcal{M}$ is a morphism and we have the relation (2), we obtain that

$$(\theta; S; h)_s(l) = (\theta; S; h)_s(r). \tag{4}$$

We notice that $h(S(C[\theta_s(l)])) = (S;h)(C[\theta_s(l)]) = (S;h)^z(C)[(S;h)(\theta_s(l))] = (S;h)^z(C)[(\theta_s;S;h)(l)] = (S;h)^z(C)[(\theta_s;S;h)(r)] = (S;h)^z(C)[(S;h)(\theta_s(r))] = (S;h)(C[\theta_s(r)])$.

From (3) it follows that $h(S(C[\theta_s(l)])) \in \Delta_M$. Hence $S(C[\theta_s(l)]) \in \equiv_\Gamma^{\mathcal{B}}$. □

Proposition 5.3. *The rule of extended paramodulation is correct.*

Proof. It follows from Propositions 4.3, 5.1 and 5.2. □

Corollary 5.1. *The rule of paramodulation is correct.*

6. Rewriting and pararewriting

For each set of formal equations Q in $\mathcal{A}$, its <u>context closure</u> is

$$\longrightarrow_Q = \{c[a] \doteq c[b] \mid a \doteq b \text{ in } Q,\ c \text{ context}\}$$

Remark that its reflexive and transitive closure $\xrightarrow{*}_Q$ is the least reflexive, transitive and context closed relation which includes Q.

$a \downarrow_Q b$ means that there exists c such that $a \xrightarrow{*}_Q c$ and $b \xrightarrow{*}_Q c$.

We fix Γ a set of conditional equations. The <u>*rewriting relation*</u> $\overset{*}{\Longrightarrow}_\Gamma$ in a fix algebra $\mathcal{A}$ may be defined as the least relation which is reflexive, transitive, compatible with the algebra operations and closed under the rewriting rule:

Rew$_\Gamma$: *for any* $(\forall X)l \doteq_s r$ if $H \in \Gamma$ *and any* $h : T_\Sigma(X) \to \mathcal{A}$, *if for any* $u \doteq_t v \in H$ *there exists* $a_{uv} \in A$ *such that* $h_t(u) \doteq_t a_{uv}$ *and* $h_t(v) \doteq_t a_{uv}$, *then* $h_s(l) \doteq_s h_s(r)$.

Some proves are easier to give if we have another definition of $\overset{*}{\Longrightarrow}_\Gamma$ as $\xrightarrow{*}_Q$, where Q is defined as follows:

$$\begin{aligned} Q_0 &= \emptyset \\ Q_{n+1} &= \{h_s(l) \doteq h_s(r)) \mid (\forall Y)l \doteq r \text{ if } H \in \Gamma,\ h : T_\Sigma(Y) \to \mathcal{A} \\ \text{and } (\forall u &\doteq v \in H) h(u) \downarrow_{Q_n} h(v)\} \\ Q &= \bigcup_{n \geq 0} Q_n \end{aligned}$$

A proof that the two definitions are equivalent is given in [2].

By definition $a \downarrow_\Gamma b \Leftrightarrow (\exists c)\ a \overset{*}{\Longrightarrow}_\Gamma c$ and $b \overset{*}{\Longrightarrow}_\Gamma c$.

We recall that Δ stands for a set of true equalities. We observe that the identity of $T_\Sigma(Y)$ is a solution for $(\exists Y)\Delta$, because $\Gamma \models (\forall Y)\Delta$. Therefore, if $G \xrightarrow{*}_p \Delta$ with the calculated morphism σ, then σ is a solution for $(\exists X)G$.

Theorem 6.1. *If $a \downarrow_\Gamma d$, then $\{a \doteq_s d\} \xrightarrow{*}_{pr} \Delta$.*

Proof. Suppose that $a \downarrow_\Gamma d$. Then there exists v such that $a \overset{*}{\Longrightarrow}_\Gamma v$ and $d \overset{*}{\Longrightarrow}_\Gamma v$. From the definition of $\overset{*}{\Longrightarrow}_\Gamma$, we have $a \xrightarrow{*}_Q v$ and $d \xrightarrow{*}_Q v$. Because $Q = \bigcup_{n\in N} Q_n$ and $Q_0 \subseteq Q_1 \subseteq \ldots \subseteq Q_n \subseteq \ldots$, there exists n such that $a \xrightarrow{*}_{Q_n} v$ and $d \xrightarrow{*}_{Q_n} v$. Therefore $a \downarrow_{Q_n} d$.

We proof by induction on n that $\{a \doteq_s d\} \xrightarrow{*}_{pr} \Delta$.

The case $n = 0$ is obvious. Suppose that for any x, y if $x \downarrow_{Q_n} y$, then $\{x \doteq_s y\} \xrightarrow{*}_{pr} \Delta$. Suppose that $a \downarrow_{Q_{n+1}} d$.

At this point, we make another induction on the number of steps $\longrightarrow_{Q_{n+1}}$ used.

If the number of steps is 0, then $a = d$ and the rest is obvious.

Otherwise, suppose, for example, that $a \longrightarrow_{Q_{n+1}} w$ and $w \downarrow_{Q_{n+1}} d$ with a step less. By the induction hypothesis, we get that $\{w \doteq_s d\} \xrightarrow{*}_{pr} \Delta$.

As $a \rightarrow_{Q_{n+1}} w$, there exists $(\forall Y)l \doteq_{s'} r$ if $H \in \Gamma$, the Σ-morphism $h : T_\Sigma(Y) \rightarrow T_\Sigma(X)$ such that $h(u) \downarrow_{Q_n} h(v)$, for each $u \doteq v \in H$, and the context c in $T_\Sigma(X \cup \{z\})$ with the properties $a = c[h_{s'}(l)]$ and $w = c[h_{s'}(r)]$. We remark that

$$\{c[h_{s'}(l)] \doteq_s d\} \longrightarrow_{pr} h(H) \cup \{c[h_{s'}(r)] \doteq_s d\}.$$

Therefore, because $\{w \doteq_s d\} \xrightarrow{*}_{pr} \Delta$, we infer that $\{a \doteq_s d\} \xrightarrow{*}_{pr} h(H) \cup \Delta$. Because $h(u) \downarrow_{Q_n} h(v)$, for any $u \doteq v \in H$, by the induction hypothesis, we get $h(H) \xrightarrow{*}_{pr} \Delta$, hence $\{a \doteq_s d\} \xrightarrow{*}_{pr} \Delta$. □

Corollary 6.1. *If G is a finite set such that $G \subseteq \downarrow_\Gamma$, then $G \xrightarrow{*}_{pr} \Delta$.*

7. Completeness of paramodulation

We recall that $\equiv_\Gamma \supseteq \downarrow_\Gamma$. When the semantic congruence $\equiv_\Gamma$ and $\downarrow_\Gamma$ are equal, $\downarrow_\Gamma$ is said to be complete. It is known that if $\overset{*}{\Longrightarrow}_\Gamma$ is confluent, then $\downarrow_\Gamma$ is complete.

Let us suppose that Γ is a set of conditional equations so that the followings are fulfilled:
$(\forall x)x \doteq x \in \Gamma$, for any variable x, $(\forall x_1 \forall x_2 \ldots \forall x_n) f(x_1, x_2, \ldots, x_n) \doteq f(x_1, x_2, \ldots, x_n) \in \Gamma$, for any operation symbol f, and, for any axiom $(\forall Y)l \doteq_s r$ if $H \in \Gamma$, any variable from Y appears in l.

Theorem 7.1. (The completeness of paramodulation) *On the previous conditions, if $\downarrow_\Gamma$ is complete, then any solution can be obtained by the rule of paramodulation only.*

Proof. Let $\sigma : T_\Sigma(X) \rightarrow T_\Sigma(Y)$ be a solution for $(\exists X)G$, i.e. $\Gamma \models (\forall Y)\sigma(G)$. Therefore $\sigma(G) \subseteq \equiv_\Gamma$.

Because $\downarrow_\Gamma$ is complete, we get $\downarrow_\Gamma = \equiv_\Gamma$. Hence $\sigma(G) \subseteq \downarrow_\Gamma$. Using Corollary we get $\sigma(G) \xrightarrow{*}_{pr} \Delta$.

Because for any axiom $(\forall Y) l \doteq_s r$ if $H \in \Gamma$, any variable of Y appears in l, by Proposition 4.2, we obtain that any pararewriting is a particular case of paramodulation with the calculated morphism the identity. Therefore $\sigma(G) \xrightarrow{*}_p \Delta$ with the calculated morphism the identity.

By the substitution lemma, we infer that $G \xrightarrow{*}_p \sigma(G)$, with the calculated morphism σ.

So $G \xrightarrow{*}_p \Delta$ with the calculated morphism σ. □

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