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# Invariant-Set-Based Analysis and Design – A Survey of Some Noticeable Contributions

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**Abstract.** Invariant-set-based techniques represent a research direction in systems engineering emerged during the last two decades. The authors of the current paper have brought some noticeable contributions to the development of this direction. The exposition of our results is connected to the international evolution of the field. Our material offers an overview structured at two levels:

1. Present framework – results available for invariant sets with general shapes described by arbitrary Hölder $p$-norms: (i) Types of dynamical systems and invariant sets under consideration; (ii) Invariance and stability; (iii) Invariance criteria for nonlinear systems; (iv) Invariance criteria for linear systems (time-variant, time-invariant, positive, with interval type uncertainties); (v) Linear synthesis based on invariant sets; (vi) Comparison methods for invariant sets.

2. Researches prefiguring the present framework – results for invariant sets with rectangular shapes (i) Linear time-invariant systems; (ii) Linear systems with interval-type uncertainties; (iii) Linear synthesis; (iv) Nonlinear systems.

## 1. Introduction

The exploration of invariant sets with respect to system dynamics was initiated by (Nagumo, 1942) and developed between 1960 and 1980 by well-known mathematicians such as (Yorke, 1968) (Crandall, 1972), (Martin Jr, 1973), (Brezis, 1976), (Pavel, 1977) – see monograph [1]. In the mid nineties, this topic has also been addressed by control engineering specialists researchers such as (Voicu, 1984), (Bitsoris, 1988), (Molchanov and Pyatnitskii, 1986), (Blanchini, 1990) – see monograph [2]. The investigations in the area of systems theory and engineering increased during the tenth decade, the significant results being discussed by the survey paper [3]. A more elaborated version of this survey with ample extensions, including numerous examples and covering the evolution till 2005 inclusive yielded the monograph [2].

The current paper presents the contributions of the three authors to the development of system analysis and design techniques based on invariant sets. Chronologically speaking, these contributions initially focused on invariant sets with rectangular shapes and, later on, the results referred to invariant sets with general shapes described by arbitrary Hölder $p$-norms. Unlike most of the approaches reported in literature, which consider constant or exponentially decreasing invariant sets, our researches encompass the general case of sets with arbitrary time dependence.

To ensure a unified exposition and for brevity reasons as well, our presentation does not follow chronological terms. Thus, Section 2 provides a picture of the present framework we have built relying on our entire research experience in the field, most of the results being fairly recent and treating invariant sets with arbitrary forms. The previous contributions, limited to invariant sets with rectangular shapes, are discussed in Section 3 that also accommodates these earlier results as particular cases of the general construction developed by Section 2. This strategy in the organization of the paper makes Sections 2 and 3 unbalanced from the material-allocation point of view, but it ensures a global vision on the currently available tools corroborated with the progress of our researches during the past two decades. For extended approaches (including proofs of the results, examples, comparisons with other techniques etc.) the reader is guided to the full texts of our works.

Throughout the paper we use the following notations:

• For a vector $\mathbf{x} \in \mathbb{R}^n$, $||\mathbf{x}||_p$ is the Hölder vector $p$-norm defined by $||\mathbf{x}||_p = [|x_1|^p + \ldots + |x_n|^p]^{1/p}$ for $1 \leq p < \infty$, and by $||\mathbf{x}||_\infty = \max_{1 \leq i \leq n} |x_i|$ for $p = \infty$.

• For a matrix $\mathbf{M} \in \mathbb{R}^{n \times n}$, $||\mathbf{M}||_p$ is the matrix norm induced by the vector $p$ -norm $|| \bullet ||_p$, $1 \leq p \leq \infty$; $\mu_{||\ ||_p}(\mathbf{M}) = \lim_{\theta \downarrow 0} \frac{1}{\theta} [||\mathbf{I} + \theta \mathbf{M}||_p - 1]$ is a matrix measure (also called "logarithmic norm") based on the matrix norm $|| \bullet ||_p$.

If $\mathbf{M} \in \mathbb{R}^{n \times n}$ is a symmetrical matrix, $\mathbf{M} \prec 0$ ($\mathbf{M} \prec= 0$) means that matrix $\mathbf{M}$ is negative definite (semidefinite). If $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{n \times m}$, then "$\mathbf{X} \leq \mathbf{Y}$", "$\mathbf{X} < \mathbf{Y}$" mean componentwise inequalities.

## 2. Present framework – results available for invariant sets with general shapes

### 2.1. Types of dynamical systems and invariant sets considered by our researches

The present framework provides concepts and instruments for studying invariance properties relative to *dynamical systems* described by

$$\dot{\mathbf{x}}(t) = \mathbf{f}\left(\mathbf{x}(t), t\right), \quad \mathbf{x}(t_0) = \mathbf{x}_0, \quad t \geq t_0, \tag{1}$$

where $\mathbf{f} : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ is continuously differentiable in $x \in \mathbb{R}^n$ and continuous in $t \in \mathbb{R}_+$; $\mathbf{f}(0, t) = 0$ for all $t \in \mathbb{R}_+$, meaning that the origin $\{0\}$ is an *equilibrium.* Let $\mathbf{x}(t\,; t_0, \mathbf{x}_0)$ denote the state-space trajectory of system (1) initialized in $\mathbf{x}(t_0) = \mathbf{x}_0$.

Two types of sets are considered, each of them being described by Hölder $p$-norms, $1 \leq p \leq \infty$.

Sets with *arbitrary time-dependence*, defined by

$$S^c_{p,\mathbf{H}(t)} = \left\{ \mathbf{x} \in \mathbb{R}^n | \, ||\mathbf{H}^{-1}(t)\mathbf{x}||_p \leq c \right\}, \ \ t \geq 0, \ c > 0, \tag{2}$$

where $\mathbf{H}(t)$ is a diagonal matrix whose diagonal entries are positive functions, continuously differentiable:

$$\mathbf{H}(t) = \operatorname{diag}\left\{h_1(t), \cdots, h_n(t)\right\}, \quad h_i(t) > 0, \ t \in \mathbb{R}_+, \ \ i = 1, \ldots, n. \tag{3}$$

Sets with *exponential time-dependence*, defined by

$$S^c_{p,\mathbf{D}e^{rt}} = \left\{ \mathbf{x} \in \mathbb{R}^n | \, ||\mathbf{D}^{-1}\mathbf{x}||_p \leq ce^{rt} \right\}, \ \ t \geq 0, \ c > 0, \tag{4}$$

where $\mathbf{D}$ is a diagonal matrix whose diagonal entries are positive constants:

$$\mathbf{D} = \operatorname{diag}\left\{d_1, \ldots, d_n\right\}, \quad d_i > 0, \ \ i = 1, \ldots, n, \tag{5}$$

and $r < 0$ is a negative constant.

In geometrical terms, the aforementioned sets posses the following characteristics (that also suggest their significance for applications):

The axes of the coordinate system in $\mathbb{R}^n$ represent *symmetry axes*, regardless of the Hölder $p$-norm, $1 \leq p \leq \infty$.

The Hölder $p$-norm defines the *shape* of the set at each time $t \geq 0$. For the frequently-used Hölder $p$-norms with $p \in \{1, 2, infty\}$, the shape is an hyper-diamond, an hyper-ellipsis, or an hyper-rectangle (regarded as generalizations of the representations in $\mathbb{R}^2$).

For a given (but arbitrary constant) $c > 0$, the *lengths* of the $n$ semi-axes at a given moment $t \geq 0$, are defined by $ch_i(t) > 0$ for $S^c_{p,\mathbf{H}(t)}$ (2) and $cd_ie^{rt} > 0$ for $S^c_{p,\mathbf{D}e^{rt}}$ (4).

**Definition 1.** (*Invariance of a set of form $S^c_{p,\mathbf{H}(t)}$ / $S^c_{p,\mathbf{D}e^{rt}}$ with respect to a system*)

Let $1 \leq p \leq \infty$ and $c > 0$. The set $S^c_{p,\mathbf{H}(t)}$ / $S^c_{p,\mathbf{D}e^{rt}}$ defined by (2) / (4) is *flow invariant with respect to* (abbreviated as *FI w.r.t.*) system (1), if any trajectory $\mathbf{x}(t; t_0, \mathbf{x}_0)$ of system (1) initialized at $t_0$ in $S^c_{p,\mathbf{H}(t_0)}/S^c_{p,\mathbf{D}e^{rt_0}}$ does not leave $S^c_{p,\mathbf{H}(t)}$ / $S^c_{p,\mathbf{D}e^{rt}}$ for any $t \geq t_0$, i.e.

(a) for the set $S^c_{p,\mathbf{H}(t)}$:

$$\begin{array}{c} \forall t_0 \in \mathbb{R}_+, \forall \mathbf{x}_0 \in \mathbb{R}^n, \ ||\mathbf{H}^{-1}(t_0)\mathbf{x}_0||_p \leq c \implies \\ \forall t > t_0, \ ||\mathbf{H}^{-1}(t)\mathbf{x}(t; t_0, \mathbf{x}_0)||_p \leq c\,; \end{array} \tag{6}$$

(b) for the set $S^{\,c}_{p,\mathbf{D}e^{rt}}$:

$$\begin{aligned}&\forall t_0 \in \mathbb{R}_+, \forall x_0 \in \mathbb{R}^n,\ \|\mathbf{D}^{-1}\mathbf{x}_0\|_p \le ce^{rt_0} \Longrightarrow \\ &\quad \forall t > t_0,\ \|\mathbf{D}^{-1}\mathbf{x}(t;t_0,\mathbf{x}_0)\|_p \le ce^{rt}\,.\end{aligned} \tag{7}$$

□

**Remark 1.** (*Comments on the equilibrium trajectories considered for the dynamical systems described by* (1))

As mentioned in the first paragraph of the current subsection, our work considers the state-space origin $\{0\}$ as an equilibrium trajectory for the nonlinear system of form (1). Generally speaking, nonlinear systems may also exhibit equilibrium trajectories defined by sets with an infinite number of points (such as closed orbits, or limit cycles). This case was not addressed by our research yet, and requires a more complex scenario. For instance, if $\mathcal{X}_e \subset \mathbb{R}^n$ denotes the set of all points corresponding to an equilibrium trajectory, then the time-dependent sets of form (2) are to be replaced by

$$\tilde{S}^{\,c}_{p,\mathbf{H}(t)} = \bigcup_{\tilde{\mathbf{x}}\in\mathcal{X}_e} \left\{ \mathbf{x} \in \mathbb{R}^n \,|\, \|\mathbf{H}^{-1}(t)(\mathbf{x}-\tilde{\mathbf{x}})\|_p \le c \right\},\ t \ge 0,\ c > 0. \tag{2'}$$

The development of invariance criteria for this type of sets cannot be approached by a direct generalization of the results presented below for (2) and, for the time being, it may be regarded as a problem remained open for further investigations. □

### 2.2. Set invariance versus stability

This subsection analyzes the connections between the invariant sets and the properties of stability, asymptotic stability or exponential stability of the equilibrium $\{0\}$ of system (1). The local or global character of the property is taken into consideration. We show that the existence of sets FI w.r.t. system (1) yields a refinement of the stability concepts.

**Theorem 1.** [4] (*Connection with the "stability" property*)

Let $1 \le p \le \infty$. Assume that the functions $h_i(t)$, $i = 1,\ldots,n$, in (3) are bounded.

**(i)** If there exists $\rho > 0$ such that $\forall c \in (0,\rho]$, the sets $S^{\,c}_{p,\mathbf{H}(t)}$ are FI w.r.t. system (1), then equilibrium $\{0\}$ of system (1) is *locally stable*. If system (1) is autonomous (time-invariant), then equilibrium $\{0\}$ of system (1) is *locally uniformly stable*.

**(ii)** If $\forall c > 0$ the sets $S^{\,c}_{p,\mathbf{H}(t)}$ are FI w.r.t. system (1), then equilibrium $\{0\}$ of system (1) is *globally stable*. If system (1) is autonomous (time-invariant), then equilibrium $\{0\}$ of (1) is *globally uniformly stable*. □

**Theorem 2.** [4] (*Connection with the "asymptotic stability" property*)

Let $1 \le p \le \infty$. Assume the functions $h_i(t)$, $i = 1,\ldots,n$, in (3) satisfy the conditions

$$\lim_{t\to\infty} h_i(t) = 0, \quad i = 1,\ldots,n. \tag{8}$$

**(i)** If there exists $\rho > 0$ such that $\forall c \in (0,\rho]$, the sets $S^{\,c}_{p,\mathbf{H}(t)}$ are FI w.r.t. system (1), then equilibrium $\{0\}$ of system (1) is *locally asymptotically stable*. If system (1) is autonomous (time-invariant), then equilibrium $\{0\}$ of system (1) is *locally uniformly asymptotically stable.*

**(ii)** If $\forall c > 0$ the sets $S^c_{p,\mathbf{H}(t)}$ are FI w.r.t. system (1), then equilibrium $\{0\}$ of system (1) is *globally asymptotically stable.* If system (1) is autonomous (time-invariant), then equilibrium $\{0\}$ of system (1) is *globally uniformly asymptotically stable.* □

**Theorem 3.** [4] (*Connection with the "exponential stability" property*)
Let $1 \le p \le \infty$.

**(i)** If there exists $\rho > 0$ such that $\forall c \in (0, \rho]$, the sets $S^c_{p,\mathbf{D}e^{rt}}$ are FI w.r.t. system (1), then equilibrium $\{0\}$ of system (1) is *locally exponentially stable.*

**(ii)** If $\forall c > 0$ the sets $S^c_{p,\mathbf{D}e^{rt}}$ are FI w.r.t. system (1), then equilibrium $\{0\}$ of system (1) is *globally exponentially stable.* □

**Remark 2.** (*Refinement of some stability concepts*)
The converse parts of Theorems 1 – 3 are false in general, showing that the invariance properties are stronger than the stability ones. Consequently, relying on the invariance properties, we get the following refinement for the stability concepts of equilibrium $\{0\}$ of (1):

(a) *Diagonally invariant stability relative to the p-norm* (abbreviated $\mathrm{DIS}_p$) local or global – if there exists $\mathbf{H}(t)$ for which Theorem 1 is satisfied.

(b) *Diagonally invariant asymptotic stability relative to the p-norm* (abbreviated $\mathrm{DIAS}_p$) local or global – if there exists $\mathbf{H}(t)$ for which Theorem 2 is satisfied.

(c) *Diagonally invariant exponential stability relative to the p-norm* (abbreviated $\mathrm{DIES}_p$) local or global – if there exists $\mathbf{D}e^{rt}$ for which Theorem 3 is satisfied. □

### 2.3. Invariance criteria for nonlinear systems

This subsection provides sufficient conditions for the invariance of the sets of form $S^c_{p,\mathbf{H}(t)}$ / $S^c_{p,\mathbf{D}e^{rt}}$ with respect to nonlinear, time-variant systems. The time-invariant case represents a particular form of the results.

Consider the Jacobian matrix with respect to $x$ of the vector function $\boldsymbol{f}$ defining system (1)

$$\mathbf{J}(\mathbf{x}, t) = [\partial \mathbf{f}(\mathbf{x}, t)/\partial \mathbf{x}] \in \mathbb{R}^{n \times n}. \tag{9}$$

Also consider the $n \times n$ matrix defined by:

$$\mathbf{A}(\mathbf{x}, t) = \int_0^1 \mathbf{J}(s\mathbf{x}, t)\mathrm{d}s, \tag{10}$$

that allows writing system (1) in the equivalent form

$$\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{x}(t), t)\,\mathbf{x}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0, \quad t \ge t_0. \tag{1'}$$

**Theorem 4.** [5] (*Invariance of the sets of form* $S^c_{p,\mathbf{H}(t)}$)
Let $1 \le p \le \infty$.

**(i)** Let $\rho > 0$ a positive constant and $\Omega_p \subseteq \mathbb{R}^n$ a set with the property $S^{\rho}_{p,\mathbf{H}(t)} \subseteq \Omega_p$, $\forall t \in \mathbb{R}_+$. (For instance, if functions $h_i(t)$, $i = 1, \ldots, n$, in (3) are bounded, then $\Omega_p$ can be defined as $\Omega_p = \{\mathbf{x} \in \mathbb{R}^n |\ ||\mathbf{H}^{-1}_{\sup}\mathbf{x}||_p \le \rho\}$ with $\mathbf{H}_{\sup} = \mathrm{diag}\{\sup_{t\in\mathbb{R}_+} h_1(t), \cdots, \sup_{t\in\mathbb{R}_+} h_n(t)\}$). The sets $S^{c}_{p,\mathbf{H}(t)}$, $c \in (0, \rho]$, are FI w.r.t. system (1), if one of the following conditions is fulfilled:

$$\mu_{||\ ||_p}\left(\mathbf{H}^{-1}(t)\mathbf{A}(\mathbf{x}, t)\mathbf{H}(t) - \mathbf{H}^{-1}(t)\dot{\mathbf{H}}(t)\right) \le 0, \tag{11a}$$

or

$$\mu_{||\ ||_p}\left(\mathbf{H}^{-1}(t)\mathbf{J}(\mathbf{x}, t)\mathbf{H}(t) - \mathbf{H}^{-1}(t)\dot{\mathbf{H}}(t)\right) \le 0, \tag{11b}$$

for any $t \in \mathbb{R}_+$ and any $\mathbf{x} \in \Omega_p$.

**(ii)** The sets $S^{c}_{p,\mathbf{H}(t)}$, $c > 0$, are FI w.r.t. system (1), if one of the conditions (11a) or (11b) is fulfilled for any $t \in \mathbb{R}_+$ and any $\mathbf{x} \in \mathbb{R}^n$. □

**Theorem 5.** [5] (*Invariance of the sets of form* $S^{c}_{p,\mathbf{D}e^{rt}}$)
Let $1 \le p \le \infty$.

**(i)** Let $\rho > 0$ a positive constant and $\Omega_p \subseteq \mathbb{R}^n$ a set with the property $S^{\rho}_{p,\mathbf{D}e^{rt}} \subseteq \Omega_p$ for $\forall t \in \mathbb{R}_+$. (For instance, $\Omega_p$ can be defined as $\Omega_p = \{\mathbf{x} \in \mathbb{R}^n |\ ||\mathbf{D}^{-1}\mathbf{x}||_p \le \rho\}$). The sets $S^{c}_{p,\mathbf{D}e^{rt}}$, $c \in (0, \rho]$, are FI w.r.t. system (1), if one of the following conditions is fulfilled for $\forall t \in \mathbb{R}_+$ and $\forall \mathbf{x} \in \Omega_p$ :

$$\mu_{||\ ||_p}\left(\mathbf{D}^{-1}\mathbf{A}(\mathbf{x}, t)\mathbf{D}\right) \le r, \tag{12a}$$

or

$$\mu_{||\ ||_p}\left(\mathbf{D}^{-1}\mathbf{J}(\mathbf{x}, t)\mathbf{D}\right) \le r. \tag{12b}$$

**(ii)** The sets $S^{c}_{p,\mathbf{D}e^{rt}}$, $c > 0$, are FI w.r.t. system (1), if one of the conditions (12a) or (12b) is fulfilled for $\forall t \in \mathbb{R}_+$ and $\forall \mathbf{x} \in \mathbb{R}^n$. □

### 2.4. Invariance criteria for linear systems

This subsection provides sufficient conditions for the invariance of the sets of form $S^{c}_{p,\mathbf{H}(t)}$ / $S^{c}_{p,\mathbf{D}e^{rt}}$ with respect to the following types of linear systems: time-variant, time-invariant, positive, interval.

*2.4.1. Linear time-variant systems*

Consider the linear time-variant system

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0, \quad t \ge t_0, \tag{13}$$

where $\mathbf{A}(t)$ is an $n \times n$ matrix whose entries are continuous functions for $t \in \mathbb{R}_+$.

**Theorem 6.** [5] (*Invariance of the sets of form* $S^{c}_{p,\mathbf{H}(t)}$)

Let $1 \le p \le \infty$. The sets $S^{\,c}_{p,\mathbf{H}(t)}$, $c > 0$, are FI w.r.t. system (13), if and only if the following condition is fulfilled:

$$\forall t \in \mathbb{R}_+, \quad \mu_{\|\ \|_p} \Big( \mathbf{H}^{-1}(t)\mathbf{A}(t)\mathbf{H}(t) - \mathbf{H}^{-1}(t)\dot{\mathbf{H}}(t) \Big) \le 0. \tag{14}$$

□

**Theorem 7.** [5] (*Invariance of the sets of form* $S^{\,c}_{p,\mathbf{D}e^{rt}}$)
Let $1 \le p \le \infty$. The sets $S^{\,c}_{p,\mathbf{D}e^{rt}}$, $c > 0$, are FI w.r.t. system (13), if and only if the following condition is fulfilled:

$$\forall t \in \mathbb{R}_+, \quad \mu_{\|\ \|_p} \left( \mathbf{D}^{-1}\mathbf{A}(t)\mathbf{D} \right) \le r. \tag{15}$$

□

*2.4.2. Linear time-invariant systems*

Consider the linear time-invariant system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0, \quad t \ge t_0, \tag{16}$$

where $\mathbf{A} \in \mathbb{R}^{n \times n}$ is a matrix with constant entries.

**Remark 3.** (*Consequence of Theorems 6 and 7*)
Let $1 \le p \le \infty$. The sets $S^{\,c}_{p,\mathbf{H}(t)}/S^{\,c}_{p,\mathbf{D}e^{rt}}$, $c > 0$, are FI w.r.t. system (16), if and only if the condition (14) / (15) is fulfilled for $\mathbf{A}(t) = \mathbf{A}$ (the constant matrix defined by (16)). □

**Remark 4.** [6], [7] (*Generalization of the "diagonal stability" concept*)
Condition (15) written for the constant matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ in the form

$$\mu_{\|\ \|_p} \left( \mathbf{D}^{-1}\mathbf{A}\mathbf{D} \right) < 0 \tag{17}$$

represents the generalization for an arbitrary Hölder norm of the well-known Lyapunov inequality $\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} \prec 0$ with $\mathbf{P} = (\mathbf{D}^{-1})^2$ [8], that is equivalent to $\mu_{\|\ \|_2} \left( \mathbf{D}^{-1}\mathbf{A}\mathbf{D} \right) < 0$. In other words, inequality (17) characterizes the diagonal stability relative to a Hölder $p$-norm of the matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$. By developing this new point of view, we have shown that the diagonal stability is not a concept exclusively associated with the quadratic norm ($p = 2$), as treated before our investigations in [7]. □

Given a matrix $\mathbf{A} = (a_{ij}) \in \mathbb{R}^{n \times n}$, define the "bar" operator ($^-$) that provides the matrix $\bar{\mathbf{A}} = (\bar{a}_{ij})$ built as follows:

$$\bar{a}_{ii} = a_{ii}, i = 1, \dots, n\ ;\ \ \bar{a}_{ij} = |a_{ij}|, i \ne j. \tag{18}$$

**Theorem 8.** [7] (*Existence of the sets of form* $S^{\,c}_{p,\mathbf{D}e^{rt}}$ *FI w.r.t. system* (16))
**(a)** Let $p = 1, \infty$. There exist sets $S^{\,c}_{p,\mathbf{D}e^{rt}}$, $c > 0$, FI w.r.t. system (16), if and only if the matrix $\bar{\mathbf{A}}$ is Hurwitz stable.

**(b)** Let $1 < p < \infty$. There exist sets $S^c_{p,\mathbf{D}e^{rt}}$, $c > 0$, FI w.r.t. system (16), if the matrix $\bar{\mathbf{A}}$ is Hurwitz stable. □

*2.4.3. Linear positive systems*

Consider the linear positive system defined by (16) where $\mathbf{A}$ is an essentially non-negative matrix.

**Theorem 9.** [9] (*Existence of the sets of form $S^c_{p,\mathbf{D}e^{rt}}$ FI w.r.t. the linear positive system*)

Let $1 \le p \le \infty$. There exist sets $S^c_{p,\mathbf{D}e^{rt}}$, $c > 0$, FI w.r.t. positive system (16), if and only if the matrix $\bar{\mathbf{A}}$ is Hurwitz stable. □

*2.4.4. Linear systems with interval type uncertainties*

Consider the linear system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbb{R}^n, \quad t \ge t_0, \quad \mathbf{A} \in \mathcal{A}^I, \tag{19}$$

where $\mathcal{A}^I$ is an interval matrix

$$\mathcal{A}^I = \{\mathbf{A} \in \mathbb{R}^{n\times n} |\ \mathbf{A}^- \le \mathbf{A} \le \mathbf{A}^+\}, \tag{20}$$

defined by the componentwise inequalities $a^-_{ij} \le a_{ij} \le a^+_{ij}$, $i,j = 1,\dots,n$, with $a^-_{ij}$, $a_{ij}$, $a^+_{ij}$ denoting the generic elements of the matrices $\mathbf{A}^-$, $\mathbf{A}$, $\mathbf{A}^+$. Define the majorant matrix of $A^I$, denoted by $\mathbf{U} = (u_{ij})_{i,j=1,\dots,n}$, built as follows:

$$\begin{aligned} u_{ii} &= \sup_{\mathbf{A}\in\mathcal{A}^I} \{a_{ii}\} = a^+_{ii}\ , \quad i = 1,\dots,n, \\ u_{ij} &= \sup_{\mathbf{A}\in\mathcal{A}^I} \{|a_{ij}|\} = \max\{|a^-_{ij}|\ ,\ |a^+_{ij}|\}\ , \quad i \ne j\ , \quad i,j = 1,\dots,n. \end{aligned} \tag{21}$$

**Theorem 10.** [10] (*Invariance of the sets of form $S^c_{p,\mathbf{H}(t)}$*)

**(a)** Let $p = 1,\infty$. The sets $S^c_{p,\mathbf{H}(t)}$, $c > 0$, are FI w.r.t. interval system (19) if and only if the following condition is fulfilled:

$$\forall t \in \mathbb{R}_+, \quad \mu_{\|\ \|_p}\left(\mathbf{H}^{-1}(t)\mathbf{U}\mathbf{H}(t) - \mathbf{H}^{-1}(t)\dot{\mathbf{H}}(t)\right) \le 0. \tag{22}$$

**(b)** Let $1 < p < \infty$. The sets $S^c_{p,\mathbf{H}(t)}$, $c > 0$, are FI w.r.t. interval system (19) if condition (22) is fulfilled. □

**Theorem 11.** [10] (*Invariance of the sets of form $S^c_{p,\mathbf{D}e^{rt}}$*)

**(a)** Let $p = 1,\infty$. The sets $S^c_{p,\mathbf{D}e^{rt}}$, $c > 0$, are FI w.r.t. interval system (19) if and only if the following condition is fulfilled:

$$\mu_{\|\ \|_p}\left(\mathbf{D}^{-1}\mathbf{U}\mathbf{D}\right) \le r \tag{23}$$

**(b)** Let $1 < p < \infty$. The sets $S^c_{p,\mathbf{D}e^{rt}}$, $c > 0$, are FI w.r.t. interval system (19) if condition (23) is fulfilled. □

**Theorem 12.** [10] (*Existence of the sets of form $S^c_{p,\mathbf{D}e^{rt}}$ FI w.r.t. the interval system* (19))

**(a)** Let $p = 1, \infty$. There exist sets $S^c_{p,\mathbf{D}e^{rt}}$, $c > 0$, FI w.r.t. interval system (19) if and only if the matrix $\mathbf{U}$ is Hurwitz stable.

**(b)** Let $1 < p < \infty$ There exist sets $S^c_{p,\mathbf{D}e^{rt}}$, $c > 0$, FI w.r.t. interval system (19) if the matrix $\mathbf{U}$ is Hurwitz stable. □

**Remark 5.** [10] (*Necessity for Theorems 11 – 12, part* (b))

Part (b) of Theorems 11 – 12 also represents a necessary condition if there exists a matrix $\mathbf{A}^* \in \mathcal{A}^I$ with the property $\mu_{\|\ \|_p}\left(\mathbf{D}^{-1}\mathbf{A}^*\mathbf{D}\right) = \mu_{\|\ \|_p}\left(\mathbf{D}^{-1}\mathbf{U}\mathbf{D}\right)$ (for instance if $\mathbf{U} \in \mathcal{A}^I$). □

### 2.5. Linear synthesis based on invariant sets

This subsection exploits set invariance for designing: • state-feedback laws for linear systems, which keep the closed-loop trajectories within sets of form $S^c_{p,\mathbf{D}e^{rt}}$, • state-variable observers for linear systems, which ensure the componentwise monitoring of the estimation error, by keeping the error trajectories within sets of form $S^c_{\infty,\mathbf{D}e^{rt}}$. The design procedures are numerically tractable.

Consider the linear system

$$\begin{array}{ll} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), & \mathbf{x}(t_0) = \mathbf{x}_0, \quad t \geq t_0, \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), & \\ \mathbf{A} \in \mathbb{R}^{n\times n}, \mathbf{B} \in \mathbb{R}^{n\times m}, \mathbf{C} \in \mathbb{R}^{p\times n}. & \end{array} \tag{24}$$

**Theorem 13.** [11] (*FI of the sets of form $S^c_{p,\mathbf{D}e^{rt}}$ w.r.t. the state-feedback closed-loop system*)

Let $1 \leq p \leq \infty$. There exists a state feedback

$$\mathbf{u}(t) = \mathbf{K}\mathbf{x}(t), \ \ \mathbf{K} \in \mathbb{R}^{m\times n} \tag{25}$$

that ensures the invariance of the sets $S^c_{p,\mathbf{D}e^{rt}}$, $c > 0$, with respect to the closed-loop system, if and only if the following condition is fulfilled:

$$\mu_{\|\ \|_p}\left(\mathbf{D}^{-1}(\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{D}\right) \leq r. \tag{26}$$

□

**Theorem 14.** [11] (*State feedback design for the usual p-norms, $p \in \{1, 2, \infty\}$*)

**(i)** For $p = 1$, condition (26) is equivalent to the following linear inequalities:

$$\begin{array}{l} -\mathbf{B}\mathbf{K} - \mathbf{G} \leq -\mathbf{A}, \\ (\mathbf{B}\mathbf{K} - \mathbf{G})^{off} \leq (\mathbf{A})^{off}, \\ \mathbf{G}^T\delta \leq r\delta, \end{array} \tag{27}$$

where $\delta = [\delta_1 \dots \delta_n]^T \in \mathbb{R}^n$, $\delta_i = 1/d_i$, $i = 1, \dots, n$, $(*)^{off}$ is a matrix with null diagonal entries and the off-diagonal elements are taken from the matrix *, whereas $\mathbf{K} \in \mathbb{R}^{m\times n}$, $\mathbf{G} \in \mathbb{R}^{n\times n}$ are unknown matrices.

**(ii)** For $p = 2$, condition (26) is equivalent to the following linear matrix inequality:

$$(\mathbf{A} - \mathbf{BK})\mathbf{D}^2 + \mathbf{D}^2(\mathbf{A} - \mathbf{BK})^T - 2r\mathbf{D}^2 \prec= 0, \tag{28}$$

where $\mathbf{K} \in \mathbb{R}^{m\times n}$ is an unknown matrix.

**(iii)** For $p = \infty$, condition (26) is equivalent to the following linear inequalities:

$$\begin{array}{l} -\mathbf{BK} - \mathbf{G} \leq -\mathbf{A}, \\ (\mathbf{BK} - \mathbf{G})^{off} \leq (\mathbf{A})^{off}, \\ \mathbf{Gd} \leq r\mathbf{d}, \end{array} \tag{29}$$

where $\mathbf{d} = [d_1 \dots d_n]^T \in \mathbb{R}^n$, $i = 1, \dots, n$, $(*)^{off}$ is a matrix with null diagonal entries and the off-diagonal elements are taken from the matrix *, whereas $\mathbf{K} \in \mathbb{R}^{m\times n}$, $\mathbf{G} \in \mathbb{R}^{n\times n}$ are unknown matrices. □

**Remark 6.** (*Numerical tractability of Theorem 14*)
If $p = 1$ or $p = \infty$, the resolution of inequalities (27) or (29) can be approached as a linear programming problem. If $p = 2$, inequality (28) is handled as an LMI [12]. Each of the three procedures operates as a computable necessary and sufficient condition, in the sense that either provides a state feedback (25), or guarantees that such a feedback law does not exist [11]. □

**Remark 7.** (*State-feedback design for interval systems and* $p = \infty$)
Consider system (24) where $\mathbf{A} \in \mathcal{A}^I$ and $\mathbf{B} \in \mathcal{B}^I$, with $\mathcal{A}^I$ and $\mathcal{B}^I$ interval matrices. Paper [13] formulates numerically tractable necessary and sufficient conditions for the existence of the state feedback (25) that ensures the invariance of sets $S^c_{p,\mathbf{D}e^{rt}}$, $p = \infty$, $c > 0$, with respect to the closed-loop system. This approach represents a generalization of Theorem 14 (iii) for the case of interval matrices. □

**Remark 8.** [14] (*Observer design with componentwise monitored error*)
The dynamics of the estimation-error vector $x_e(t)$ is described by

$$\dot{\mathbf{x}}_e(t) = (\mathbf{A} - \mathbf{LC})\mathbf{x}_e(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0, \quad t \geq t_0. \tag{30}$$

The invariance of the rectangular sets $S^c_{\infty,\mathbf{D}e^{rt}}$, $c > 0$, that allow monitoring each component of the estimation-error vector, is ensured by the necessary and sufficient condition (similar to Theorem 13):

$$\mu_{\|\ \|_\infty}\left(\mathbf{D}^{-1}(\mathbf{A} - \mathbf{LC})\mathbf{D}\right) \leq r. \tag{31}$$

Condition (31) is equivalent to the following linear inequalities (similar to Theorem 14 (iii)):

$$\begin{array}{l} -\mathbf{LC} - \mathbf{G} \leq -\mathbf{A}, \\ (\mathbf{LC} - \mathbf{G})^{off} \leq (\mathbf{A})^{off}, \\ \mathbf{Gd} \leq r\mathbf{d}. \end{array} \tag{32}$$

□

### 2.6. Comparison methods for invariant sets

This subsection applies the comparison theory for deriving results adequate to set invariance.

Consider a time-variant nonlinear system, of form (1) and a linear positive system of form (16), whose matrix is denoted by $\Gamma$ :

$$\dot{\mathbf{x}}(t) = \Gamma \mathbf{x}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0, \quad t \geq t_0, \tag{16'}$$

In the following, system (16') is used as a comparison system.

For the matrix-valued functions $\mathbf{J}(\mathbf{x}, t)$ (9) and $\mathbf{A}(\mathbf{x}, t)$ (10) associated with system (1), use the "bar" operator $(\bar{\ })$ defined by (18) in order to build the matrix-valued functions $\overline{\mathbf{J}(\mathbf{x}, t)}$ and $\overline{\mathbf{A}(\mathbf{x}, t)}$.

**Theorem 15.** [5] (*Sets of form $S^{\,c}_{p,\mathbf{H}(t)}$ / $S^{\,c}_{p,\mathbf{D}e^{rt}}$ FI w.r.t. nonlinear systems*) Let $1 \leq p \leq \infty$.

**(i)** Let $\rho > 0$ a positive constant and $\Omega_p \subseteq \mathbb{R}^n$ a set with the property $S^{\,\rho}_{p,\mathbf{H}(t)}$ / $S^{\,\rho}_{p,\mathbf{D}e^{rt}} \subseteq \Omega_p$, $\forall t \in \mathbb{R}_+$. If one of the following two conditions:

$$\overline{\mathbf{A}(\mathbf{x}, t)} \leq \Gamma, \tag{33a}$$

or

$$\overline{\mathbf{J}(x, t)} \leq \Gamma, \tag{33b}$$

is fulfilled for $\forall t \in \mathbb{R}_+$, $\forall \mathbf{x} \in \Omega_p$, and the sets $S^{\,c}_{p,\mathbf{H}(t)}/\,S^{\,c}_{p,\mathbf{D}e^{rt}}$, $c > 0$, are FI w.r.t. the comparison system (16'), then the sets $S^{\,c}_{p,\mathbf{H}(t)}/\,S^{\,c}_{p,\mathbf{D}e^{rt}}$, $c \in (0, \rho]$, are FI w.r.t. the nonlinear system (1).

**(ii)** If one of the conditions (33a) or (33b) is fulfilled $\forall t \in \mathbb{R}_+$, $\forall \mathbf{x} \in \mathbb{R}^n$, and the sets $S^{\,c}_{p,\mathbf{H}(t)}/\,S^{\,c}_{p,\mathbf{D}e^{rt}}$, $c > 0$, are FI w.r.t. the comparison system (16'), then the sets $S^{\,c}_{p,\mathbf{H}(t)}/\,S^{\,c}_{p,\mathbf{D}e^{rt}}$, $c > 0$, are FI w.r.t. the nonlinear system (1). □

**Remark 9.** (*Sets of form $S^{\,c}_{p,\mathbf{H}(t)}/\,S^{\,c}_{p,\mathbf{D}e^{rt}}$ FI w.r.t. recurrent neural networks*) The dynamics of a recurrent neural network is described with respect to the equilibrium $\{0\}$ by:

$$\dot{\mathbf{x}}(t) = \mathbf{B}\mathbf{x}(t) + \mathbf{W}\mathbf{g}(\mathbf{x}(t)), \tag{34}$$

where $\mathbf{x} = [x_1, \cdots, x_n]^T \in \mathbb{R}^n$, $\mathbf{B}, \mathbf{W} \in \mathbb{R}^{n \times n}$ with $\mathbf{B} = \mathrm{diag}\{b_1, \cdots, b_n\}$, $b_i < 0$, $i = 1, \ldots, n$. The vector-valued function $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^n$, $\mathbf{g}(\mathbf{x}) = [g_1(\mathbf{x}) \cdots g_n(\mathbf{x})]^T$, is continuously differentiable on $\mathbb{R}^n$, satisfying the conditions $g_i(\mathbf{x}) = g_i(x_i)$, $g_i(0) = 0$ and $0 \leq g_i'(s) \leq L_i$, $\forall s \in \mathbb{R}$, $i = 1, \ldots, n$. Let $\Pi = \mathbf{B} + \widetilde{\mathbf{W}}\Lambda$, where the matrix $\widetilde{\mathbf{W}} = [\tilde{w}_{ij}] \in \mathbb{R}^{n \times n}$ has the elements $\tilde{w}_{ii} = \max\{0, w_{ii}\}$, $i = 1, \ldots, n$, $\tilde{w}_{ij} = |w_{ij}|$, $i \neq j$, $i, j = 1, \ldots, n$, and $\Lambda = \mathrm{diag}\{L_i, \cdots, L_n\}$ is a diagonal matrix. Theorem 15 allows studying the invariance of the sets $S^{\,c}_{p,\mathbf{H}(t)}/\,S^{\,c}_{p,\mathbf{D}e^{rt}}$, $c > 0$, with respect to the neural network (34), by using the linear positive system

$$\dot{\mathbf{x}}(t) = \Pi\, \mathbf{x}(t), \tag{35}$$

as a comparison system. □

## 3. Researches prefiguring the present framework – results for invariant sets with rectangular shapes

The construction of the present framework relied on our previous investigations of invariant rectangular sets of type $S^c_{p,\mathbf{H}(t)}/\, S^c_{p,\mathbf{D}e^{rt}}$ with $p = \infty$. These investigations are reviewed by the survey paper [15]. At that time, the rectangular sets were written in the particular form of hyper-intervals $[-ch_1(t), ch_1(t)] \times \ldots \times [-ch_n(t), ch_n(t)]$ / $[-cd_1e^{rt}, cd_1e^{rt}] \times \ldots \times [-cd_ne^{rt}, cd_ne^{rt}]$, that later on proved to be equivalent with norm-based description (2) / (4) for $p = \infty$.

The current section just recalls those results that represented key elements for the development of our approaches, permitting the extension from sets defined by the norm $\infty$ to sets described by arbitrary Hölder norms.

### 3.1. Linear time-invariant systems

Papers [16] and [17] were devoted to systems of form (16) and marked the beginning of the researches on invariant sets of rectangular type. They brought the following contributions: • Proofs of Theorems 6 and 7 for $\mathbf{A}$ a constant matrix and $p = \infty$, the inequalities (14) and (15) being obtained in particular forms $\dot{\mathbf{h}}(t) \geq \bar{\mathbf{A}}\mathbf{h}(t)$, $\mathbf{h}(t) = [h_1(t) \cdots h_n(t)]^T$ and $r\mathbf{d} \geq \bar{\mathbf{A}}\mathbf{d}$, $\mathbf{d} = [d_1 \cdots d_n]^T$, respectively, where $\bar{\mathbf{A}}$ is defined by (18). • Proofs of Theorem 8 for $p = \infty$. • Refinement of the stability concepts by introducing the *componentwise asymptotic stability* (abbreviated CWAS) and the *componentwise exponential asymptotic stability* (abbreviated CWEAS), which correspond to $\mathrm{DIAS}_p$ and $\mathrm{DIES}_p$ for $p = \infty$ in subsection 2.2.

Paper [18] completed the approach in [19] that considered rectangular sets nonsymmetrical with respect to $\{0\}$. It proved that CWAS and CWEAS testing in the nonsymmetrical case can be reduced to the Hurwitz stability analysis of the matrix $\bar{\mathbf{A}}$, exactly as in the symmetrical case.

### 3.2. Linear systems with interval-type uncertainties

Papers [20], [21] proved Theorems 10 – 12 for $p = \infty$. Paper [21] also treated the case of rectangular sets nonsymmetrical with respect to $\{0\}$. It proved that CWAS and CWEAS testing in the nonsymmetrical case can be reduced to the Hurwitz stability analysis of the matrix $\mathbf{U}$, exactly as in the symmetrical case.

### 3.3. Linear synthesis

Papers [22], [23] addressed the state-feedback synthesis that ensures the CWEAS property for the closed-loop system. They used inequality (26) with $p = \infty$ in the particular form $r\mathbf{d} \geq \overline{(\mathbf{A}-\mathbf{BK})}\,\mathbf{d}$ with $\mathbf{d} = [d_1 \cdots d_n]^T$, but the proposed algorithm is more restrictive than the resolution of inequalities (29).

Paper [24] approached the observer design with componentwise monitored error. It used inequality (31) with $p = \infty$ in the particular form $r\mathbf{d} \geq \overline{(\mathbf{A}-\mathbf{LC})}\,\mathbf{d}$ with $\mathbf{d} = [d_1 \cdots d_n]^T$, and formulated necessary and sufficient conditions for the analytical computation of matrix $\mathbf{L}$.

### 3.4. Nonlinear systems

Paper [25] considered state-space representations of the form (1') and proved Theorem 15 (i), (ii) case (a) for sets of form $S^c_{p,\mathbf{D}e^{rt}}$ with $p = \infty$.

Papers [26], [27] considered recurrent neural networks of type (34) with uncertainties and provided tests for the existence of invariant sets of form $S^c_{p,\mathbf{H}(t)}/\,S^c_{p,\mathbf{D}e^{rt}}$, $c > 0$, with $p = \infty$. Uncertainties in [26] referred to the slopes of the functions $g_i(x_i)$, whereas the uncertainties in [27] took into account both the slopes of functions $g_i(x_i)$ and the values of the entries of matrices $B$ and $W$ (considered as interval matrices). The proposed testing strategies rely on the proof of Theorem 15 (ii) case (a) for $p = \infty$.

## 4. Concluding remarks

Relying on the exploration of invariant sets with rectangular shapes that started in the mid eighties, we have foreseen that most of the investigated properties could remain valid for invariant sets with general shapes. It took some time to mathematically formalize this intuition, but we have managed to develop a nice generalization which accommodates our earlier researches on rectangular sets as a particular case.

The new scenario provides analysis and synthesis tools for sets with arbitrary or exponential time-dependence, described by Hölder $p$-norms, $1 \leq p \leq \infty$. Besides the practical role in applications, the present framework has an important theoretical value, by proving the existence of a unified theory for many problems previously treated as completely independent one to the other.

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