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Nominal Event Structures

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Abstract. We present the concept of "event structures" in the nominal framework of the Fraenkel-Mostowski model of set theory. Using specific techniques of nominal logic, we introduce and study the *nominal event structures*, providing some properties. The analogy between the results obtained by using the Fraenkel-Mostowski axioms and those obtained by using the Zermelo-Fraenkel axioms is discussed.

1. Introduction

The notion of *fresh name* often arises when manipulating syntactic expressions; therefore it is necessary to indicate some constraints whenever describing such a syntactic manipulation. Often it is just said that a name is *fresh* without specifying any restrictions. In such a case, we mean that the fresh name must be different from any name occurring anywhere else in the current expression or program. Some programming systems have explicit mechanisms for renaming, for binding a name with a value, and for managing sets of such bindings. Modern programming languages are designed to manage bindings and fresh names by using the notions of scope, workspace and environments. Since renaming, binding and fresh names appear in several approaches, they deserve to be studied in their own terms.

Nominal logic and nominal semantics was presented by Gabbay and Pitts in [2, 3, 5]; they use the Fraenkel-Mostowski (FM) model of set theory. The FM permutation model of set theory was devised in 1930s to prove the independence of the axiom of choice from the other axioms of Zermelo-Fraenkel (ZF) model of set theory. The axiom of choice says that given any collection of sets, each containing at least one object it is possible to make a selection of exactly one object from each set, even though there are an infinite amount of sets and no "rule" of how we choose objects [7]. As shown by Gödel and Cohen, the axiom of choice is proved to be logically independent of the other axioms of ZF set theory. The FM model is built using all the axioms of the

Zermelo-Fraenkel with atoms (ZFA) model, except the axiom of choice. It has the special property of finite support which claims that for each element x in an arbitrary FM-set we can find a finite set supporting x (in the sense of Definition 2.4), and the property that the set A of atoms is infinite. In fact, the finite support property says that for each element x in an arbitrary FM-set, we can always find a fresh element for x *i.e.* an element which is not in the support of x (see Corollary 2.1 for the formal definition of support).

In λ-calculus, a general computation model in computer science, α-equivalence classes of a λ-term x have the support of x represented by the free variables with respect to x (Example 2.2). This means that Fraenkel-Mostowski model of set theory could be a more suitable framework for computer science.

The aim of this paper is to define and study "elementary event structures", "event structures" and "causality function" in nominal framework of FM-sets. These concepts were initially described using the classical ZF model of set theory [4]. Using nominal techniques, we define "FM-elementary event structures", "FM-event structures" and "FM-causality function", presenting also some properties of these new concepts. The analogy between the results obtained by using the FM axioms of set theory and those obtained by using the ZF axioms of set theory is emphasized by the results presented in Section 3.

2. Fraenkel-Mostowski Sets

We present the Fraenkel-Mostowski model with atoms by using the notions of transposition, permutation, and substitution. We start from the ZFA model. Let A be an infinite set of atoms. A is characterized by the axiom "$y \in x \Rightarrow x \notin A$" which means that only non-atoms can have elements.

Definition 2.1.

i) A *transposition* is a function $(a\,b) : A \to A$ with the property $(a\,b)(a) = b$, $(a\,b)(b) = a$ and $(a\,b)(n) = n$ for $n \neq a, b$.

ii) A *permutation* of A is a bijection π from A to A.

iii) A *substitution* is a function $\{b|a\} : A \to A$ with the property $\{b|a\}(n) = n$ if $n \neq a$ and $\{b|a\}(a) = b$.

Let S_A be the set of all finitary permutations of A (*i.e.* the set of all permutations of A which leave unchanged all but finitely many atoms). S_A is a group under the usual composition of permutations. The composition of permutation is denoted by "$\circ$". It can be proved that S_A is the set of all bijections $\pi : A \to A$ generated by composing finitely many transpositions. Indeed, let $\sigma \in S_A$ be a function which permutes only a finite number of atoms $\{a_1, ..., a_n\}$ such that the atoms $A \setminus \{a_1, ..., a_n\}$ are left unchanged. Formally, we can say that σ is a permutation of the set $\{a_1, ..., a_n\}$, and so σ can be expressed as a product of at most $(n-1)$ transpositions [6].

The underlying logic of ZFA is the usual first-order logic with equality; its signature contains just a binary predicate set membership $\in$ and a constant symbol A.

Definition 2.2. The following axioms give a complete characterization of the Zermelo-Fraenkel with atoms model:

1. $\forall x.(\exists y.y \in x) \Rightarrow x \notin A$ (only non-atoms can have elements)
2. $\forall x, y.(x \notin A \,\text{and y} \notin \text{A and } \forall \text{z.(z} \in \text{x} \Leftrightarrow \text{z} \in \text{y))} \Rightarrow \text{x} = \text{y}$ (axiom of extensionality)
3. $\forall x, y.\exists z.z = \{x, y\}$ (axiom of pairing)
4. $\forall x.\exists y.y = \{z \,|\, z \subset x\}$ (axiom of powerset)
5. $\forall x.\exists y.y \notin A \,\text{and}\, \text{y} = \{\text{z} \,|\, \exists \text{w.(z} \in \text{w and w} \in \text{x)}\}$ (axiom of union)
6. $\forall x.\exists y.(y \notin A \,\text{and}\, \text{y} = \{\text{f(z)} \,|\, \text{z} \in \text{x}\})$, for each functional formula $f(z)$ (axiom of replacement)
7. $\forall x.\exists y.(y \notin A \,\text{and}\, \text{y} = \{\text{z} \,|\, \text{z} \in \text{x and p(z)}\})$, for each formula $p(z)$ (axiom of separation)
8. $(\forall x.(\forall y \in x.p(y)) \Rightarrow p(x)) \Rightarrow \forall x.p(x)$ (induction principle)
9. $\exists x.(\emptyset \in x \,\text{and}\, (\forall \text{y.y} \in \text{x} \Rightarrow \text{y} \cup \{\text{y}\} \in \text{x}))$ (axiom of infinite)
10. $A\,\text{is not finite.}$

Definition 2.3. Let X be a set defined by the axioms of ZFA model. An S_A-*action* on X is a function $\cdot : S_A \times X \to X$ having the properties that $Id \cdot x = x$ and $\pi \cdot \pi^{'} \cdot x = (\pi \circ \pi^{'}) \cdot x$ for all $\pi, \pi' \in S_A$ and $x \in X$.

An S_A-*set* is a pair $(X, \cdot)$ where X is a set defined by ZFA model, and $\cdot : S_A \times X \to X$ is an S_A-action on X. We simply use X if no confusion arises.

Definition 2.4. Let $(X, \cdot)$ be an S_A-set. We say that $S \subset A$ *supports* x whenever for each $\pi \in Fix(S)$ we have $\pi \cdot x{=}x$, where $Fix(S){=}\{\pi \,|\, \pi(a) = a, \forall a \in S\}$.

When we can find a finite set supporting an element x in an S_A-set, we say that "x has the finite support property" or "x is finitely supported".

Definition 2.5. Let $(X, \cdot)$ be an S_A-set. We say that X is an *FM-set* if for each $x \in X$ there exists a finite set $S_x \subset A$ which supports x.

Theorem 2.1. *Let X be an S_A-set, and for each $x \in X$ let us define $\mathcal{F}_x = \{S \subset A \,|\, S\, finite,\, S\, supports\, x\}$. If $\mathcal{F}_x$ is nonempty then it has a least element which also supports x. We call this element the support of x, and we denote it by $S(x)$ or $supp(x)$.*

Proof. We define $S(x){=}\cap\{S \subset A \,|\, S\, finite,\, S\, supports\, x\}{=}\bigcap_{S\in\mathcal{F}_x} S$. We have to prove that if S_1 and S_2 are both finite and supports x, then $S_1 \cap S_2$ supports x. Indeed, let π be a permutation from $Fix(S_1 \cap S_2)$, and prove that $\pi \cdot x{=}x$. Since any permutation π is generated by composing finitely many transpositions, it is enough to prove the finite support property of x only for transpositions. This means that we should prove that $(a\,b)\cdot x{=}x$ for each $a, b \notin S_1 \cap S_2$. The cases $a, b \notin S_1$ and $a, b \notin S_2$ are obvious because S_1 and respectively S_2 supports x, and by Definition 2.4 we have $(a\,b)\cdot x{=}x$. Now let $a \notin S_1$ and $b \notin S_2$. Since $S_1 \cup S_2$ is finite and A is infinite, we can find $c \in A \setminus (S_1 \cup S_2)$ and $a \neq c \neq b$. Since $a, c \notin S_1$ and S_1 supports x it follows that $(c\,a)\cdot x{=}x$. Because $b, c \notin S_2$ and S_2 supports x, we have $(c\,b)\cdot x{=}x$. It follows that $(a\,b)\cdot x{=}(a\,b)\cdot(c\,b)\cdot x{=}((a\,b)\circ(c\,b))\cdot x{=}((c\,b)\circ(a\,c))\cdot x{=}(c\,b)\cdot(a\,c)\cdot x{=}x$. The case when $a \notin S_2$ and $b \notin S_1$ is similar. Let us suppose that $\mathcal{F}_x$ is nonempty. This means there is at least a finite set supporting x; thus, the support $S(x)$ is well defined. Moreover, $S(x)$ is minimal between the finite sets supporting x. □

Corollary 2.1. *Let X be an FM-set, and for each $x \in X$ we define $\mathcal{F}_x{=}\{S \subset A \,|\, S\, finite,\, S\, supports\, x\}$. Then $\mathcal{F}_x$ has a least element which also supports x. We call this element the support of x, and denote it by $S(x)$ or $supp(x)$.*

Example 2.1.

1. The set A of atoms is an S_A-set with the S_A-action $\cdot : S_A \times X \to X$ defined by $\pi \cdot a := \pi(a)$ for all $\pi \in S_A$ and $a \in A$. $(A, \cdot)$ is an FM-set because for each $a \in A$ we have that $\{a\}$ supports a. Moreover $S(a) = \{a\}$ for each $a \in A$.

2. The set A of atoms is an S_A-set with the S_A-action $\cdot : S_A \times X \to X$ defined by $\pi \cdot a{=}a$ for all $\pi \in S_A$ and $a \in A$. $(A, \cdot)$ is an FM-set because for each $a \in A$ we have that $\emptyset$ supports a. Moreover, $S(a){=}\emptyset$ for each $a \in A$.

3. The set S_A is an S_A-set with the S_A-action $\cdot : S_A \times S_A \to S_A$ defined by $\pi \cdot \sigma := \pi \circ \sigma \circ \pi^{-1}$ for all $\pi, \sigma \in S_A$. $(S_A, \cdot)$ is an FM-set because for each $\sigma \in S_A$ we have that the finite set $\{a \in A \,|\, \sigma(a) \neq a\}$ supports σ. Moreover $S(\sigma) = \{a \in A \,|\, \sigma(a) \neq a\}$ for each $\sigma \in S_A$.

4. Any ordinary ZF-set X (like $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$ for example) is an S_A-set with the S_A-action $\cdot : S_A \times X \to X$ defined by $\pi \cdot x := x$ for all $\pi \in S_A$ and $x \in X$. Also X is an FM-set because for each $x \in X$ we have that $\emptyset$ supports x. Moreover $S(x) = \emptyset$ for each $x \in X$.

5. The ZFA universe (*i.e.* the class of all ZFA sets), denoted by $ZFA(A)$, construct under the usual von Neumann cumulative hierarchy [3] is an S_A-set with the S_A-action $\cdot : S_A \times ZFA(A) \to ZFA(A)$ defined inductively by $\pi \cdot a := \pi(a)$ for all atoms $a \in A$ and $\pi \cdot x := \{\pi \cdot y \,|\, y \in x\}$. The FM universe (*i.e.* the class of all FM-sets), denoted by $FM(A)$, construct under the usual von Neumann cumulative hierarchy form an FM-set under the same action [3].

6. If $(X, \cdot)$ is an S_A-set then $\wp(X) = \{Y \,|\, Y \subseteq X\}$ is also an S_A-set with the S_A-action $\star : S_A \times \wp(X) \to \wp(X)$ defined by $\pi \star Y := \{\pi \cdot y \,|\, y \in Y\}$ for all permutations π of A, and all subsets Y of X. When no confusion arises we denote $\star$ also by $\cdot$. Note that $\wp(X)$ does not necessarily be an FM-set, even if X is an FM-set. For example A is an FM-set, but $\wp(A)$ is not an FM-set

because the subsets of A which are in the same time infinite and coinfinite have not the finite support property. For each FM-set $(X, \cdot)$ we denote by $\wp_{fs}(X)$ the set formed from those subsets of X which are finitely supported according to the action $\star$. $(\wp_{fs}(X), \star|_{\wp_{fs}(X)})$ is an FM-set, where $\star|_{\wp_{fs}(X)} : S_A \times \wp_{fs}(X) \to \wp_{fs}(X)$ is defined by $\pi \star|_{\wp_{fs}(X)} Y = \pi \star Y$ for all $\pi \in S_A$ and $Y \in \wp_{fs}(X)$; the codomain of the action $\star|_{\wp_{fs}(X)}$ (which is in fact the action $\star$ restricted to $\wp_{fs}(X)$) is indeed included in $\wp_{fs}(X)$ because of Proposition 2.1.

7. Let $(X, \cdot)$ and $(Y, \diamond)$ be S_A-sets. As in the classical ZF theory we define the cartesian product $X \times Y$ as the set of ordered pair $(x, y) = \{\{x\}, \{x, y\}\}$ for $x \in X$ and $y \in Y$. $X \times Y$ is also an S_A-set with the S_A-action $\star : S_A \times (X \times Y) \to (X \times Y)$ defined by $\pi \star (x, y) = (\pi \cdot x, \pi \diamond y)$ for all $\pi \in S_A$ and all $x \in X$, $y \in Y$. If $(X, \cdot)$ and $(Y, \diamond)$ are FM-sets then $(X \times Y, \star)$ is also an FM-set.

Using the remarks presented in item 6 of Example 2.1, we can define the notion of finitely supported subset of a certain FM-set.

Definition 2.6. Let $(X, \cdot)$ be an FM-set. A subset Z of X is called *finitely supported* if and only if there exists a finite set $S \subseteq A$ such that S supports Z with respect the S_A-action $\star : S_A \times \wp(X) \to \wp(X)$ defined by $\pi \star Y = \{\pi \cdot y \,|\, y \in Y\}$ for all permutations π of A and all subsets Y of X. Whenever S supports Z with respect the S_A-action $\star$, we say that *S supports Z*.

Remark 2.1. Let $(X, \cdot)$ be an FM-set. A subset Z of X is called *finitely supported* in the sense of Definition 2.6 if and only if $Z \in \wp_{fs}(X)$.

Proposition 2.1. *Let $(X, \cdot)$ be an S_A-set and let $\pi \in S_A$ be an arbitrary permutation. Then for each $x \in X$ which is finitely supported, we have that $\pi \cdot x$ is finitely supported and*

$$S(\pi \cdot x) = \pi(S(x)).$$

Proof. Let $\pi \in S_A$ be an arbitrary permutation and $x \in X$ a finitely supported element. First we show that $\pi(S(x))$ supports $\pi \cdot x$. Let $\sigma \in Fix(\pi(S(x)))$. This means that $\sigma(\pi(a)) = \pi(a)$ for all $a \in S(x)$. Also $\pi^{-1}(\sigma(\pi(a))) = \pi^{-1}(\pi(a)) = a$, for all $a \in S(x)$. Thus we get $\pi^{-1} \circ \sigma \circ \pi \in Fix(S(x))$. Now $S(x)$ always supports x (by Corollary 2.1). According to Definition 2.4, we have $(\pi^{-1} \circ \sigma \circ \pi) \cdot x = x$. Since $\cdot$ is a group action, the last equality is equivalent with $\sigma \cdot (\pi \cdot x) = \pi \cdot x$. Hence whenever $x \in X$ is finitely supported we have that $\pi \cdot x$ is finitely supported. Moreover $S(\pi \cdot x) \subseteq \pi(S(x))$ for each $x \in X$ which is finitely supported and each $\pi \in S_A$. We apply this for elements $\pi^{-1} \in S_A$ and $\pi \cdot x \in X$ (which we already know that is finitely supported). We get $S(\pi^{-1} \cdot \pi \cdot x) \subseteq \pi^{-1}(S(\pi \cdot x))$. Composing with π in the last relation, we obtain $\pi(S(x)) \subseteq S(\pi \cdot x)$. □

Definition 2.7. Let $(X, \cdot)$ be an FM-set. An element $x \in X$ is called *equivariant* if it has an empty support, *i.e.* $\pi \cdot x = x$ for each $\pi \in S_A$.

The following example (considered also in [3]) shows us how we can express the λ-calculus in FM. Gabbay and Pitts used this example to argue once more why we can work in the FM approach instead of working in the classical ZF approach. Many other reasons for which the FM model can be considered as a more suitable framework for computer science can be found in [2],[3],[5].

Example 2.2.

1. If X' is the set of λ-terms t, then we inductively define an S_A-action $\star$ of S_A on X' by:
   • **variable**: $\pi \star a = \pi(a)$ whenever a is a variable (corresponding to atoms) and π is a permutation of atoms;
   • **application**: $\pi \star (tt') = (\pi \star t)(\pi \star t')$ for all λ-terms t and t', and for all $\pi \in S_A$;
   • **abstraction**: $\pi \star (\lambda a.t) = \lambda(\pi(a)).(\pi \star t)$ for all variables a, all λ-terms t, and for all $\pi \in S_A$.

   It is easy to check that $(X', \star)$ is an FM-set and the support of a λ-term t is the finite set of atoms occurring in t (both free and binding occurrences).

2. Let X be the set of the α-equivalences classes of the λ-calculus terms t. We can define an S_A-action $\cdot$ of S_A on X by: $\pi \cdot [t]_\alpha = [\pi \star t]_\alpha$ for all λ-terms t and for all $\pi \in S_A$ (where $[t]_\alpha$represents the α-equivalence class of the λ-term t). If two λ-terms t and t' are α-equivalent, it is clear that $\pi \star t = \pi \star t'$ and so the action $\cdot$ is well defined. It is easy to check that $(X, \cdot)$ is an FM-set; in fact a set which is in a bijection with an inductively defined FM-set [3]. If t is chosen to be a representative of its α-equivalence class then $S(t)$ *coincides with* $fn(t)$, where $fn(t)$ is the set of free variables of t defined by λ-calculus rules [3]. An α-equivalence class of terms does not contain bound names (in the sense of the quotient of the equivalence class over them). We cannot define a function $bn : X \to \wp_{fin}(A)$ which would be able to extract exactly the bound names for each FM element t [3]. α-equivalent terms are identified in the nominal logic since two α-equivalent terms have the same set of free variables.

In the view of Example 2.1 we can define the notion of finitely supported functions. Recall that a function $f : X \to Y$ is a particular relation; more precisely, a function $f : X \to Y$ is a subset f of $X \times Y$ characterized by the property that for each $x \in X$ there is exactly one $y \in Y$ such that $(x, y) \in f$. A function f between two FM-sets X and Y is finitely supported if it is finitely supported as a subset of the cartesian product $X \times Y$ in the sense of Definition 2.6. If X and Y are FM-sets, then $X \times Y$ is an FM-set with the S_A-action defined as in Example 2.1 (7). Then we can give the following definition for finitely supported functions.

Definition 2.8. Let X and Y be FM-sets. A function $f : X \to Y$ is *finitely supported* if $f \in \wp_{fs}(X \times Y)$.

We denote by Y^X the set $\{f \subseteq X \times Y \mid \text{f is a function from the underlying set of } X \text{ to the underlying set of } Y\}$.

Proposition 2.2. *Let $(X, \cdot)$ and $(Y, \diamond)$ be FM-sets. Then Y^X is an S_A-set with the S_A-action $\star : S_A \times Y^X \to Y^X$ defined by*

$$(\pi \star f)(x) = \pi \diamond (f(\pi^{-1} \cdot x))$$

for all $\pi \in S_A$, $f \in Y^X$ *and* $x \in X$.

A function $f : X \to Y$ *is finitely supported in the sense of Definition 2.8 if and only if it is finitely supported with respect the permutation action* $\star$.

Proof. We already know that functions from X to Y are subsets of the cartesian product $X \times Y$ which is an FM-set (Example 2.1 (7)). By Example 2.1 (6) we know that $\wp(X \times Y)$ is an S_A-set and $\pi \star f = \{(\pi \cdot x, \pi \diamond y) \,|\, (x, y) \in f\}$. Hence $\pi \star f$ is a function with the domain $\pi \cdot X = X$. Moreover $(\pi \star f)(\pi \cdot x) = \pi \diamond f(x)$. Let $x' = \pi \cdot x$, so $x = \pi^{-1} \cdot x'$. We obtain $(\pi \star f)(x') = \pi \diamond (f(\pi^{-1} \cdot x'))$. The application $x \mapsto x'$ is bijective, and it follows that $(\pi \star f)(x) = \pi \diamond (f(\pi^{-1} \cdot x))$ for all $\pi \in S_A$, $f \in Y^X$ and $x \in X$. Since the action $\star$ of S_A on Y^X was defined as in Example 2.1 (6), it is clear that each function $f : X \to Y$ is finitely supported with respect the permutation action $\star$ if and only if $f \in \wp_{fs}(X \times Y)$. ☐

Proposition 2.3. *Let* $(X, \cdot)$ *and* $(Y, \diamond)$ *be FM-sets. A function* $f \in Y^X$ *is equivariant with respect the* S_A*-action* $\star : S_A \times Y^X \to Y^X$ *defined by*

$(\pi \star f)(x) = \pi \diamond (f(\pi^{-1} \cdot x))$ *for all* $\pi \in S_A$, $f \in Y^X$ *and* $x \in X$

if and only if for all $\pi \in S_A$ *and* $x \in X$ *we have* $f(\pi \cdot x) = \pi \diamond f(x)$.

Proof. Let us suppose that f is equivariant. This means $\pi \star f = f$ for all $\pi \in S_A$. Let $\pi \in S_A$ be an arbitrary permutation. From Proposition 2.2 we know that for each $x \in X$ we have $(\pi \star f)(\pi \cdot x) = \pi \diamond f(x)$. Since $\pi \star f = f$, it follows that $f(\pi \cdot x) = \pi \diamond f(x)$ for all $x \in X$. Conversely, let us suppose that for all $\pi \in S_A$ and $x \in X$ we have $f(\pi \cdot x) = \pi \diamond f(x)$. Let $\pi \in S_A$ be an arbitrary permutation. For each $x \in X$ we have $(\pi \star f)(x) = \pi \diamond (f(\pi^{-1} \cdot x)) = f(\pi \cdot (\pi^{-1} \cdot x)) = f(x)$. ☐

By now we denote the S_A-actions only by "$\cdot$" (if no confusion is possible).

3. Nominal Event Structures

We consider the event structures as they are formally presented in [4]. Namely, an *elementary event structure* is a partial order $(E, \leq)$, where E is a set of events and $\leq$ is a partial order over E which is considered as a causality relation. We adapt the notions presented in [4] in an FM framework. The definitions and results in this section are justified by the fact that in the FM framework only objects with finite support are allowed.

Definition 3.1. An *FM-elementary event structure* E is an FM-set $(E, \cdot)$ together with an equivariant partial order relation "$\leq$" on E. An FM-elementary event structure is denoted by $(E, \leq, \cdot)$ or simply by E.

A partial order relation "$\leq$" on E is a subset of the cartesian product $E \times E$; this relation is reflexive, anti-symmetric and transitive. By Definition 2.7, "$\leq$" is equivariant if it is finitely supported as a subset of the cartesian product $E \times E$ in the sense of Definition 2.6 and its support is empty. This means that "$\leq$" is equivariant iff for each pair $(e, e') \in \leq$ and each $\pi \in S_A$ we have that $\pi \cdot (e, e') \in \leq$ (where $\cdot$ represents the action of S_A on the cartesian product $E \times E$ constructed as in Example 2.1 (7)).

If we write "$(e, e') \in \leq$" as "$e \leq e'$ ", the equivariance property of $\leq$ can be expressed by

$$e \leq e' \text{ implies } \pi \cdot e \leq \pi \cdot e', \text{ whenever } \pi \in S_A.$$

Definition 3.2. An *FM-complete elementary event structure* E is an FM-elementary event structure $(E, \leq, \cdot)$ such that every finitely-supported subset $X \subseteq E$ has a least upper bound with respect the order relation $\leq$.

Theorem 3.1. *Let $(E, \leq, \cdot)$ be an FM-complete elementary event structure. Then every finitely-supported subset $X \subseteq E$ has a greatest lower bound with respect the order relation $\leq$.*

Proof. Let X be a finitely supported subset of E in the sense of Definition 2.6. Let $D = \cap\{\downarrow x \,|\, x \in X\}$, where by $\downarrow x$ we denote the set $\{y \in E \,|\, y \leq x\}$. Informally D is the set of lower bounds of X with respect the order relation $\leq$. If X is empty, we take $D = E$. First we show that D is finitely supported in the sense of Definition 2.6. If we prove this, D will have a least upper bound (denoted by $sup(D)$) because of Definition 3.2. We know that X has a finite support $S(X)$, and we show that $S(X)$ supports D. Let π be a permutation which fixes $S(X)$ pointwise (*i.e.*, $\pi \in Fix(S(X))$). Let $d \in D$ arbitrarily chosen; then $d \leq x$ for all $x \in X$. We claim that $\pi \cdot d \in D$, which is the same as saying that $\pi \cdot d$ is a lower bound of X. Indeed, let $y \in X$ be an arbitrary element of X. Since $\pi \in Fix(S(X))$ and $S(X)$ supports X in the sense of Definition 2.6 (cf. Theorem 2.1), we get $\pi \star X = X$ (where the action $\star$ of S_A on $\wp(E)$ is defined as in Example 2.1 (6)). This means that for our $y \in X$ there is an $x \in X$ such that $\pi \cdot x = y$. However $d \leq x$; because $\leq$ is equivariant, we also have $\pi \cdot d \leq \pi \cdot x = y$. Hence $\pi \cdot d \in D$. Because d is arbitrary from D, we can say that $\pi \star D \subseteq D$ whenever $\pi \in Fix(S(X))$ ($\bigstar$). We have two methods of proving that $\pi \star D = D$ (for $\pi \in Fix(S(X))$). First we remark that $\pi \in Fix(S(X))$ iff $\pi^{-1} \in Fix(S(X))$. According to ($\bigstar$), we get $\pi^{-1} \star D \subseteq D$ which means $\pi \star (\pi^{-1} \star D) \subseteq \pi \star D$ (the action $\star$ of S_A on $\wp(E)$ is defined as in Example 2.1 (6)), and $D \subseteq \pi \star D$.
Another method of showing that $\pi \star D = D$ is to use a proof by contradiction. Let us suppose that there is $\pi \in Fix(S(X))$ such that $\pi \star D \subsetneq D$. By induction, we get $\pi^n \star D \subsetneq D$ for all $n \geq 1$. However π is a finitary permutation, and so there is $k \in \mathbb{N}$ such that $\pi^k = Id$. We obtain $D \subsetneq D$, a contradiction. It follows that $\pi \star D = D$ whenever $\pi \in Fix(S(X))$, and hence $S(X)$ supports D according to Definition 2.4. Since $S(X)$ is finite, we have that $sup(D)$ exists.

We show now that $sup(D)$ is the greatest lower bound of X. If $x \in X$, then x is an upper bound of D, and so $sup(D) \leq x$. Since x was chosen arbitrarily from X, we can have $sup(D) \in D$. Since $sup(D)$ is maximal between the lower bounds of X and it is a lower bound of X, then $sup(D) = inf(X)$, where $inf(X)$ represents the greatest lower bound of X. □

Definition 3.3. Let $(E, \leq, \cdot)$ be an FM-elementary event structure. An *FM-causality function on E* is a causality preserving, finitely-supported function from E to E, *i.e.* a finitely supported function $f : E \to E$ with the property that: $e \leq e'$ implies $f(e) \leq f(e')$ for all $e, e' \in E$.

We provide a fixed point (Tarski-like) theorem adapted to the FM approach which states that whenever f is a causality function on an FM-complete elementary event structure E, there is at least one event unchanged by f.

Theorem 3.2. *Let $(E, \leq, \cdot)$ be an FM-complete elementary event structure and $f : E \to E$ a causality function on E. Then there exists an event $e \in E$ such that $f(e) = e$. Moreover, e can be chosen such that any other event e' which is invariant at the application of f (i.e., $f(e') = e'$) causes e (i.e., $e' \leq e$).*

Proof. Let $D = \{d \in E \,|\, d \leq f(d)\}$. First we prove that D is finitely-supported in the sense of Definition 2.6. We know that f is a causality function and hence it is finitely supported in the sense of Definition 2.8. We prove that $S(f)$ supports D in the sense of Definition 2.6. Let $\pi \in Fix(S(f))$, and $d \in D$ be arbitrarily chosen. Then $d \leq f(d)$, and because "$\leq$" is equivariant we also have $\pi \cdot d \leq \pi \cdot f(d)$. However, by Proposition 2.2, we have $\pi \cdot f(d) = (\pi \widetilde{\star} f)(\pi \cdot d)$, where $\widetilde{\star}$ is the S_A-action $\widetilde{\star} : S_A \times E^E \to E^E$ defined by $(\pi \widetilde{\star} f)(x) = \pi \cdot (f(\pi^{-1} \cdot x))$ for all $\pi \in S_A$, $f \in E^E$ and $x \in E$. Also, by Proposition 2.2, we know that f is finitely supported according to Definition 2.8 if and only if it is finitely supported according to the S_A-action $\widetilde{\star}$ described before. Since $\pi \in Fix(S(f))$ and $S(f)$ supports f (by Theorem 2.1) we have that $\pi \widetilde{\star} f = f$. It follows that $\pi \cdot d \leq \pi \cdot f(d) = (\pi \widetilde{\star} f)(\pi \cdot d) = f(\pi \cdot d)$ and hence $\pi \cdot d \in D$. Since d was chosen arbitrarily from D, we have $\pi \star D \subseteq D$ (where the action $\star$ of S_A on $\wp(E)$ is defined as in Example 2.1 (6) whenever $\pi \in Fix(S(f))$ (†).

As in the last part of the proof of Theorem 3.1, we have two methods of proving that $\pi \star D = D$ for $\pi \in Fix(S(f))$. First we remark that $\pi \in Fix(S(f))$ iff $\pi^{-1} \in Fix(S(f))$. Hence by (†) we get $\pi^{-1} \star D \subseteq D$ which means that $\pi \star (\pi^{-1} \star D) \subseteq \pi \star D$ (because of the definition of $\star$), and finally $D \subseteq \pi \star D$.
Another method to prove that $\pi \star D = D$ is by contradiction. Let us suppose that there is $\pi \in Fix(S(f))$ such that $\pi \star D \subsetneq D$. By induction we get $\pi^n \star D \subsetneq D$ for all $n \geq 1$. However π is a finitary permutation, and so there is $k \in \mathbb{N}$ such that $\pi^k = Id$. We obtain $D \subsetneq D$, a contradiction. It follows $\pi \star D = D$ whenever $\pi \in Fix(S(f))$, and hence $S(f)$ supports D according to Definition 2.4. Since $S(f)$ is finite, we have that $sup(D)$ exists according to Definition 3.2.

Let $e = sup(D)$. Then for each $d \in D$, we have $d \leq e$. Since f preserves the causality relation, we have $f(d) \leq f(e)$. Because $d \in D$, it follows $d \leq f(d) \leq f(e)$. Hence $d \leq f(e)$ for each $d \in D$. By the definition of a least upper bound, we have that $e \leq f(e)$ which means that $e \in D$.
However, because f is causality preserving, we have $f(x) \in D$ for each $x \in D$. Since $e \in D$, it follows that $f(e) \in D$; thus $f(e) \leq e$ because $e = sup(D)$.

Therefore we get $f(e) = e$. Whenever e' is an event such that $f(e') = e'$, it follows that $e' \in D$, and so $e' \leq e$. □

Theorem 3.3. *Let $(E, \leq, \cdot)$ be an FM-complete elementary event structure and $f : E \to E$ an equivariant causality function over E. Let P be the set of fixed points of f. Then $(P, \leq, \cdot)$ is an FM-complete elementary event structure.*

Proof. First we remark that P is non-empty because of Theorem 3.2. Since f is equivariant, it follows that for all $\pi \in S_A$ and all $x \in E$ we have $f(\pi \cdot x) = \pi \cdot f(x)$.

Whenever x is a fixed point of f, we have $f(\pi \cdot x) = \pi \cdot f(x) = \pi \cdot x$ and so $\pi \cdot x$ is also a fixed point of f. We proved that the application $\cdot|_P$ (where $\cdot|_P$ represents the restriction of the S_A-action $\cdot$ to P) has the codomain equal to P. Hence the application $\cdot|_P : S_A \times P \to P$ defined by $\pi \cdot|_P x = \pi \cdot x$ for all $\pi \in S_A$ and all $x \in P$ is an S_A-action of S_A on P (it satisfies the axioms of a group action whenever $\cdot$ does). Moreover, $(P, \cdot|_P)$ is an FM-set. The support of each element in P according to the S_A-action $\cdot|_P$ is the same with the support of that element according to the S_A-action $\cdot$. It is somehow natural to denote the action $\cdot|_P$ with $\cdot$.

Let X be an arbitrary finitely supported subset in P. We have to prove that X has a least upper bound in P. We already know that X has a least upper bound (denoted by $sup(X)$) in E because $(E, \leq, \cdot)$ is an FM-complete elementary event structure.

Let $x \in X$ be an arbitrary element. We have that $x \leq sup(X)$ and then $f(x) \leq f(sup(X))$. However X contains only fixed points of f and hence $f(x){=}x$ and $x \leq f(sup(X))$. By the definition of a least upper bound it follows that $sup(X) \leq f(sup(X))$. Now, let $y \geq sup(X)$. Because f is a causality function, we also have $f(y) \geq f(sup(X))$. We have already proved that $sup(X) \leq f(sup(X))$, and hence $f(y) \geq sup(X)$. We get that $f(y) \geq sup(X)$ whenever $y \geq sup(X)$.

Let $D = \{d \in E \,|\, f(d) \leq d \text{ and } sup(X) \leq d\}$. We claim that $S(sup(X))$ supports D in the sense of Definition 2.6, and so D is a finitely supported set. Indeed, $S(sup(X))$ exists according to Corollary 2.1 because E is an FM-set and $sup(X) \in E$. Let $\pi \in Fix(S(sup(X)))$, and $d \in D$ be arbitrarily chosen. Then $f(d) \leq d$. Because "$\leq$" is equivariant, we also have $\pi \cdot f(d) \leq \pi \cdot d$. However, by Proposition 2.2, we have $\pi \cdot f(d) = (\pi \widetilde{\star} f)(\pi \cdot d)$, where $\widetilde{\star}$ is the S_A-action $\widetilde{\star} : S_A \times E^E \to E^E$ defined by $(\pi \widetilde{\star} f)(x) = \pi \cdot (f(\pi^{-1} \cdot x))$ for all $\pi \in S_A$, $f \in E^E$ and $x \in E$. By the same Proposition 2.2, we know that f is finitely supported according to Definition 2.8 if and only if it is finitely supported according to the S_A-action $\widetilde{\star}$ described before. Since f is equivariant, we have that $\sigma \widetilde{\star} f = f$ for each $\sigma \in S_A$. It follows that $f(\pi \cdot d) = (\pi \widetilde{\star} f)(\pi \cdot d) = \pi \cdot f(d) \leq \pi \cdot d$. Since $d \in D$, we also have $sup(X) \leq d$. Therefore $\pi \cdot sup(X) \leq \pi \cdot d$. However $\pi \cdot sup(X) = sup(X)$ because $\pi \in Fix(S(sup(X)))$. Finally we obtain $sup(X) \leq \pi \cdot d$ and hence $\pi \star D \subseteq D$ whenever $\pi \in Fix(S(sup(X)))$ (the action $\star$ of S_A on $\wp(E)$ is defined as in Example 2.1 (6)). Since $\pi \in Fix(S(sup(X)))$ iff $\pi^{-1} \in Fix(S(sup(X)))$, it follows that $\pi^{-1} \star D \subseteq D$, from which $\pi \star (\pi^{-1} \star D) \subseteq \pi \star D$ (because of the definition of $\star$) and finally $D \subseteq \pi \star D$. Thus D is finitely supported, and there exists the greatest lower bound of D denoted by $inf(D)$ (Theorem 3.1).

Let $e = inf(D)$. Then for each $d \in D$, we have $e \leq d$. Since f preserves the causality relation, we have also $f(e) \leq f(d)$. Because $d \in D$, it follows $f(e) \leq f(d) \leq d$. Hence $f(e) \leq d$ for each $d \in D$. According to the definition of a greatest lower bound, we have that $f(e) \leq e$. Also, $d \geq sup(X)$ for each $d \in D$ implies $inf(D) \geq sup(X)$ which means $e \in D$. However, because f is causality preserving and because $f(y) \geq sup(X)$ whenever $y \geq sup(X)$, we have that $f(x) \in D$ for each $x \in D$. Since $e \in D$, it follows that $f(e) \in D$ and so $e \leq f(e)$ because $e = inf(D)$.

We proved that e is a fixed point of f such that $sup(X) \leq e$. Hence $e \in P$ is an upper bound for X. It remains to prove that e is the least upper bound for X in the system $(P, \leq)$. Let $e' \in P$ be another upper bound for X. Then $sup(X) \leq e'$

(since $sup(X)$ is the least upper bound for X in E and, clearly, e' is an upper bound for X in E); it follows that $e' \in D$. Since $e = inf(D)$, we get $e \leq e'$. This means $e = sup(X)$ is in $(P, \leq)$. □

Remark 3.1. In Theorem 3.3 we have required that f should be equivariant because we need P to be an FM-set. If f is not equivariant, then it would be possible to find a fixed point x of f such that $\pi \cdot x$ would not be again a fixed point of f (for a certain $\pi \in S_A$), and so the codomain of the function $\cdot|_P$ would not be P. Thus we could not prove in the FM approach a complete similar Tarski-like theorem ([8]): "Let $(E, \leq, \cdot)$ be an FM-complete elementary event structure and $f : E \to E$ a causality function on E. Let P be the set of fixed points of f. Then $(P, \leq, \cdot)$ is an FM-complete elementary event structure" . Only the equivariance of such a f ensures the existence of an FM-structure on P in the sense of Definition 2.5.

Various relations could be defined over the set E of events. In [4] it is introduced a conflict relation on E, denoted by $\#$. An *event structure* is defined in [4] as a triple $(E, \leq, \#)$ where $(E, \leq)$ is an elementary event structure, and $\#$ is a symmetrical and irreflexive relation on E (called the conflict relation) which satisfies: $\forall e_1, e_2, e_3 \in E$: $e_1 \geq e_2 \# e_3 \Rightarrow e_1 \# e_3$.

We can adapt this notion to FM framework, adding that both the set of events and the relations must have the finite support property.

Definition 3.4. An *FM-event structure* E is an FM-set $(E, \cdot)$ together with an equivariant partial order relation "$\leq$" on E, and with an equivariant symmetrical and irreflexive relation "$\#$" on E such that:

- $(E, \leq, \cdot)$ is an FM-elementary event structure.
- the relation $\#$ satisfies: $\forall e_1, e_2, e_3 \in E : e_1 \geq e_2 \# e_3 \Rightarrow e_1 \# e_3$.

Such an FM-event structure is denoted by $(E, \leq, \#, \cdot)$ or simply by E.

Some authors define the elementary event structures as event structures where no elements are in conflict; so the notion of elementary event structures is introduced after the notion of event structure was defined. This definition of elementary event structures is actually the same as the one defined in [4]. There are different ways of presenting the notion of event structure. The other definitions of event structures can also be formalized in FM in the same way we have presented in this paper, with the mention that in the FM framework we allow only finitely supported objects. Our aim was not to provide a complete study of event structures nor to present several approaches. Our goal was only to prove that a known concept (in this case event structures) often used in computer science, initially introduced by using the Zermelo-Fraenkel axioms of set theory, can also be formalized in the FM framework. We also provide some nominal properties of the new event structures, and prove that some ZF properties of it are preserved in a nominal approach.

The techniques presented here can be extended to other concepts. Several concepts like multisets, generalized multisets, various algebraic structures, etc can be formalized in FM setting in a similar way. The nominal techniques presented in this paper have already been used in [1], where the authors have provided a nominal semantics of some subcalculi of the π-calculus.

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