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On The Harmonic Index and The Girth for Graphs

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Abstract. The harmonic index of a graph G is defined as the sum of the weights $\frac{2}{d(u)+d(v)}$ of all edges uv of G, where $d(u)$ denotes the degree of a vertex u in G. In this work, we present the minimum and maximum values of the harmonic index for connected graphs with girth at least k ($k \geq 3$), and characterize the corresponding extremal graphs. Using this result, we obtain several relations between the harmonic index and the girth of a graph.

Keywords: harmonic index; girth; relation.

AMS Subject Classification: 05C07, 05C35.

1. Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The Randić index $R(G)$, proposed by Randić [13] in 1975, is defined as the sum of the weights $\frac{1}{\sqrt{d(u)d(v)}}$ over all edges uv of G, that is,

$$R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d(u)d(v)}},$$

where $d(u)$ denotes the degree of a vertex u of G. Randić [13] noticed that this index was well correlated with a variety of physico-chemical properties of alkanes: boiling point, enthalpy of formation, surface area and solubility in water, etc. Eventually, this index became one of the most successful molecular descriptors, and scores of its pharmacological and chemical applications have been reported. Mathematical properties of this descriptor have also been studied extensively (see [7, 9, 10, 14]).

In this paper, we consider a closely related variant of the Randić index, named the harmonic index. For a graph G, the harmonic index $H(G)$ is defined as

$$H(G) = \sum_{uv \in E(G)} \frac{2}{d(u) + d(v)}.$$

This topological index first appeared in [5], and it can also be viewed as a particular case of the general sum-connectivity index proposed by Zhou and Trinajstić in [23] (see also in [3, 4, 15]). Favaron, Mahéo and Saclé [6] considered the relation between the harmonic index and the eigenvalues of graphs. Zhong [19, 20], Zhong and Xu [21], Zhu, Chang and Wei [24] determined the minimum and maximum values of the harmonic index for simple connected graphs, trees, unicyclic graphs and bicyclic graphs, and characterized the corresponding extremal graphs. Chang and Zhu [1] found the minimum values of the harmonic index for graphs with minimum degree at least two and for triangle-free graphs with minimum degree at least k ($k \geq 1$), and characterized the corresponding extremal graphs. Ilić [8] and Xu [17] established some relationships between the harmonic index and several other topological indices. Yang and Hua [18] computed the harmonic indices of nanocones and triangular benzenoid graphs. See [2, 11, 12, 16, 22, 25] for more mathematical properties of this index.

In this paper, we first consider the minimum and maximum values of the harmonic index for connected graphs with girth at least k ($k \geq 3$) and characterize graphs for which these bounds are best possible, and then we obtain several relations between the harmonic index and the girth of a graph.

We conclude this section with some notation and terminology. Let G be a graph. For any vertex $v \in V(G)$, we use $N_G(v)$ (or $N(v)$ if there is no ambiguity) to denote the set of neighbors of v in G. Let $\Delta(G)$ and $\delta(G)$ denote the maximum and minimum degree of G, respectively. An isolated vertex is a vertex of degree 0, and a pendent vertex is a vertex of degree 1. An edge incident with a pendent vertex is called a pendent edge. We define $G - uv$ to be the graph obtained from G by deleting the edge $uv \in E(G)$, and $G + uv$ to be the graph that arises from G by adding an edge uv between two non-adjacent vertices u and v of G. By deleting a vertex v in G, we mean deleting the vertex v together with its incident edges. We use C_n to denote the cycle on n vertices. We write $A := B$ to rename B as A.

2. Main results

In this section, we prove the main results of this paper. First, we list some lemmas which will be used in the following arguments. For an edge $e = uv$ of a graph G, its weight is defined to be $\frac{2}{d(u) + d(v)}$. Chang and Zhu [1] and Liu [11] proved the following result.

Lemma 2.1. *Let e be an edge with maximum weight in a graph G, then*

$$H(G) > H(G - \{e\}).$$

Liu [11] also showed the following.

Lemma 2.2. *Let e be a pendent edge in a graph G, then*

$$H(G) > H(G - \{e\}).$$

In [19], Zhong mentioned the following result.

Lemma 2.3. *Let G be a connected graph with n vertices, then*

$$H(G) \leq \frac{n}{2},$$

with equality if and only if G is a regular graph.

We now consider the minimum value of the harmonic index for connected graphs with girth at least k ($k \geq 3$). For each $3 \leq k \leq n$, let C_k^n be the graph with n vertices obtained from a cycle C_k by attaching $n-k$ pendent vertices to exactly one vertex of C_k (see Fig. 1 for an illustration).

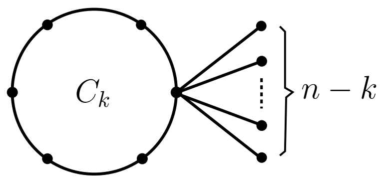

Fig. 1. The graph C_k^n.

Theorem 2.4. *Let G be a connected graph with n vertices and girth $g(G) \geq k$ ($k \geq 3$), then*

$$H(G) \geq 1 + \frac{k}{2} - \frac{6}{n-k+3} + \frac{4}{n-k+4},$$

with equality if and only if $G \cong C_k^n$.

Proof. Let m be the number of edges in G. Since G is a connected graph containing at least one cycle, we have $m \geq n$. To prove the theorem, we apply induction on $m+n$. If $n = 3$, then $m = k = 3$ and $G \cong C_3$, it is easy to check that the assertion of the theorem holds. So we may assume that $m \geq n \geq 4$ and the result holds for smaller values of $m+n$. For convenience, we may also assume that G is the extremal graph with the minimum value of the harmonic index among all connected graph with n vertices and girth at least k. We consider two cases according to the value of $\delta(G)$.

Case 1. $\delta(G) = 1$.

Let $u \in V(G)$ be a pendent vertex and let $uv \in E(G)$. Then $d(v) = d \geq 2$. Let $N(v) \setminus \{u\} = \{u_1, \ldots, u_{d-1}\}$. Since G is a connected graph containing at least one cycle, we know that there exists at least one vertex in $\{u_1, \ldots, u_{d-1}\}$ with degree at least 2. Note that $d \leq \Delta(G) \leq n - k + 2$ since $g(G) \geq k$.

Suppose there exists exactly one vertex in $\{u_1, \ldots, u_{d-1}\}$ with degree at least 2, say w. Let $d(w) = l \geq 2$ and let $N(w) \setminus \{v\} = \{v_1, \ldots, v_{l-1}\}$. Define $G' := G - \{wv_1, \ldots, wv_{l-1}\} + \{vv_1, \ldots, vv_{l-1}\}$, then G' is a connected graph with n vertices, m edges and $g(G') = g(G) \geq k$. Hence

$$\begin{aligned}
&H(G) - H(G') \\
=&\left(\sum_{i=1}^{l-1} \frac{2}{l + d(v_i)} + \frac{2(d-1)}{d+1}\right) - \left(\sum_{i=1}^{l-1} \frac{2}{d + l - 1 + d(v_i)} + \frac{2(d-1)}{d+l}\right) \\
=&\left(\sum_{i=1}^{l-1} \frac{2}{l + d(v_i)} - \sum_{i=1}^{l-1} \frac{2}{d + l - 1 + d(v_i)}\right) + (d-1)\left(\frac{2}{d+1} - \frac{2}{d+l}\right) \\
>&0,
\end{aligned}$$

since $d \geq 2$ and $l \geq 2$. But this implies that $H(G) > H(G')$, which contradicts the assumption of G.

So we conclude that there exist at least two vertices in $\{u_1, \ldots, u_{d-1}\}$ with degree at least 2 (and hence $d \geq 3$). Let $G'' := G - \{u\}$, then G'' is a connected graph with $n - 1$ vertices, $m - 1$ edges and $g(G'') = g(G) \geq k$. By induction, we have $H(G'') \geq 1 + \frac{k}{2} - \frac{6}{n-k+2} + \frac{4}{n-k+3}$. Note that $\frac{2}{d+x} - \frac{2}{d+x-1}$ is strictly monotonously increasing in $x \geq 1$ and $\frac{4}{d+2} - \frac{10}{d+1} + \frac{6}{d}$ is strictly monotonously decreasing in $3 \leq d \leq n - k + 2$, we have

$$\begin{aligned}
H(G) =& H(G'') + \frac{2}{d+1} + \sum_{i=1}^{d-1}\left(\frac{2}{d + d(u_i)} - \frac{2}{d - 1 + d(u_i)}\right) \\
\geq& H(G'') + \frac{2}{d+1} + 2 \cdot \left(\frac{2}{d+2} - \frac{2}{d+1}\right) + (d-3) \cdot \left(\frac{2}{d+1} - \frac{2}{d}\right) \\
=& H(G'') + \frac{4}{d+2} - \frac{10}{d+1} + \frac{6}{d} \\
\geq& \left(1 + \frac{k}{2} - \frac{6}{n-k+2} + \frac{4}{n-k+3}\right) + \frac{4}{d+2} - \frac{10}{d+1} + \frac{6}{d} \\
\geq& \left(1 + \frac{k}{2} - \frac{6}{n-k+2} + \frac{4}{n-k+3}\right) + \frac{4}{n-k+4} - \frac{10}{n-k+3} + \frac{6}{n-k+2} \\
=& 1 + \frac{k}{2} - \frac{6}{n-k+3} + \frac{4}{n-k+4}
\end{aligned}$$

with equalities if and only if $d(v) = d = n-k+2$, exactly two vertices in $\{u_1, \ldots, u_{d-1}\}$ have degree 2 and the other $d-3$ vertices in $\{u_1, \ldots, u_{d-1}\}$ have degree 1, i.e., $G \cong C_k^n$.

Case 2. $\delta(G) \geq 2$.

If $m = n$, then we have $k = n$ and $G \cong C_n$, it is easy to check that the assertion of the theorem holds. So we may assume that $m \geq n + 1$. Let $e^* = uv$ be an edge with maximum weight in G. We consider two subcases according to e^* is a cut-edge in G or not.

Subcase 2.1. e^* is a cut-edge in G.

Since e^* is a cut-edge, we know that $G - \{e^*\}$ has exactly two components, say G_1 and G_2, with $u \in V(G_1)$ and $v \in V(G_2)$. Let n_i be the number of vertices in G_i (for $i = 1, 2$), then $n = n_1 + n_2$. Since $\delta(G) \geq 2$ and $g(G) \geq k$, we have G_i contains at least one cycle with $g(G_i) \geq g(G) \geq k$, and hence $n_i \geq k$ (for $i = 1, 2$). By induction, we see that $H(G_i) \geq 1 + \frac{k}{2} - \frac{6}{n_i-k+3} + \frac{4}{n_i-k+4}$. Then by Lemma 2.1,

$$\begin{aligned}
&H(G) - H(C_k^n) > H(G - \{e^*\}) - H(C_k^n) = H(G_1) + H(G_2) - H(C_k^n) \\
\geq& \left(1 + \frac{k}{2} - \frac{6}{n_1 - k + 3} + \frac{4}{n_1 - k + 4}\right) + \left(1 + \frac{k}{2}\right. \\
&\left. - \frac{6}{n_2 - k + 3} + \frac{4}{n_2 - k + 4}\right) - \left(1 + \frac{k}{2} - \frac{6}{n - k + 3} + \frac{4}{n - k + 4}\right) \\
=& 1 + \frac{k}{2} - \left(\frac{4}{n_1 - k + 3} - \frac{4}{n_1 - k + 4}\right) - \frac{2}{n_1 - k + 3} \\
&- \left(\frac{4}{n_2 - k + 3} - \frac{4}{n_2 - k + 4}\right) - \frac{2}{n_2 - k + 3} + \left(\frac{6}{n - k + 3} - \frac{4}{n - k + 4}\right) \\
=& 1 + \frac{k}{2} - \frac{4}{(n_1 - k + 3)(n_1 - k + 4)} - \frac{2}{n_1 - k + 3} \\
&- \frac{4}{(n_2 - k + 3)(n_2 - k + 4)} - \frac{2}{n_2 - k + 3} + \left(\frac{6}{n - k + 3} - \frac{4}{n - k + 4}\right) \\
>& 1 + \frac{k}{2} - \frac{4}{12} - \frac{2}{3} - \frac{4}{12} - \frac{2}{3} \\
=& \frac{k}{2} - 1 > 0.
\end{aligned}$$

But this implies that $H(G) > H(C_k^n)$, which contradicts the assumption of G.

Subcase 2.2. e^* is not a cut-edge in G.

Let $G' := G - \{e^*\}$, then G' is a connected graph with n vertices, $m - 1$ edges and $g(G') \geq g(G) \geq k$. By Lemma 2.1 and by induction, we have

$$H(G) > H(G - \{e^*\}) = H(G') \geq 1 + \frac{k}{2} - \frac{6}{n - k + 3} + \frac{4}{n - k + 4} = H(C_k^n),$$

again a contradiction. This completes the proof of the theorem. □

For the maximum value of the harmonic index for connected graphs with girth at least k $(k \geq 3)$, by Lemma 2.3, we obtain the following theorem.

Theorem 2.5. *Let G be a connected graph with n vertices and girth $g(G) \geq k$ $(k \geq 3)$, then*

$$H(G) \leq \frac{n}{2},$$

with equality if and only if G is a regular graph.

Note that $1 + \frac{k}{2} - \frac{6}{n-k+3} + \frac{4}{n-k+4}$ is strictly monotonously increasing in $3 \leq k \leq n$, together with Theorems 2.4 and 2.5, we have the following relations between the harmonic index and the girth of a graph, which generalize the main result in [16] (by letting $k = 3$).

Theorem 2.6. *For any connected graph G with n vertices and girth $g(G) \geq k$ $(k \geq 3)$,*

$$1 + \frac{3k}{2} - \frac{6}{n-k+3} + \frac{4}{n-k+4} \leq H(G) + g(G) \leq \frac{3n}{2}$$

and

$$\left(1 + \frac{k}{2} - \frac{6}{n-k+3} + \frac{4}{n-k+4}\right) \cdot k \leq H(G) \cdot g(G) \leq \frac{n^2}{2}.$$

The lower bounds are attained if and only if $G \cong C_k^n$, and the upper bounds are attained if and only if $G \cong C_n$.

In fact, we can also obtain some other relations as follows.

Theorem 2.7. *For any connected graph G with n vertices and girth $g(G) \geq k$ $(k \geq 3)$,*

$$-\frac{n}{2} \leq H(G) - g(G) \leq \frac{n}{2} - k \quad \text{and} \quad \frac{1}{2} \leq \frac{H(G)}{g(G)} \leq \frac{n}{2k}.$$

The lower bounds are attained if and only if $G \cong C_n$, and the upper bounds are attained if and only if G is a regular graph with $g(G) = k$.

Proof. The upper bounds follow immediately from Theorem 2.5. We now prove the lower bounds.

If $G \cong C_n$, then we have $H(G) = \frac{n}{2}$ and $g(G) = n$, and thus the lower bounds hold. So we may assume that $G \ncong C_n$ (and hence $g(G) < n$). We construct a finite sequence $G_0, G_1, \ldots, G_s$ $(s \geq 1)$ of graphs with the following properties:

(i) $G_0 := G$;

(ii) If exactly one component of G_i is a cycle and all the other components are isolated vertices, then let $G_s := G_i$. Otherwise, let G_{i+1} be the graph obtained from G_i by deleting a pendent edge (if G_i contains at least one pendent vertex) or deleting an edge with maximum weight (if G_i contains no pendent vertices) for every $0 \leq i < s$.

Note that deleting an edge will not decrease the girth of a graph, since G_i contains at least one cycle for every $0 \le i \le s$. Then by Lemmas 2.1 and 2.2, we have

$$g(G_0) \le g(G_1) \le \cdots \le g(G_s) = 2H(G_s) < \cdots < 2H(G_1) < 2H(G_0),$$

i.e.,

$$g(G) < 2H(G).$$

Hence

$$H(G) - g(G) > \frac{g(G)}{2} - g(G) = -\frac{g(G)}{2} > -\frac{n}{2} \quad \text{and} \quad \frac{H(G)}{g(G)} > \frac{1}{2}.$$

This finishes the proof of the theorem. □

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References

[1] Chang R., Zhu Y., *On the harmonic index and the minimum degree of a graph*, Romanian J. Inf. Sci. Tech. **15** (2012) pp. 335–343.

[2] Deng H., Balachandran S., Ayyaswamy S. K., Venkatakrishnan Y. B., *On the harmonic index and the chromatic number of a graph*, Discrete Appl. Math. **161** (2013) pp. 2740–2744.

[3] Du Z., Zhou B., Trinajstić N., *Minimum general sum-connectivity index of unicyclic graphs*, J. Math. Chem. **48** (2010) pp. 697–703.

[4] Du Z., Zhou B., Trinajstić N., *On the general sum-connectivity index of trees*, Appl. Math. Lett. **24** (2011) pp. 402–405.

[5] Fajtlowicz S., *On conjectures of Graffiti-II*, Congr. Numer. **60** (1987) pp. 187–197.

[6] Favaron O., Mahéo M., Saclé J. F., *Some eigenvalue properties in graphs (conjectures of Graffiti-II)*, Discrete Math. **111** (1993) pp. 197–220.

[7] Gutman I., Furtula B. (Eds.), *Recent Results in the Theory of Randić Index*, University of Kragujevac, Kragujevac, 2008.

[8] Ilić A., *Note on the harmonic index of a graph*, `http://arxiv.org/abs/1204.3313v1`.

[9] Li X., Gutman I., *Mathematical Aspects of Randić-Type Molecular Structure Descriptors*, University of Kragujevac, Kragujevac, 2006.

[10] Li X., Shi Y., *A survey on the Randić index*, MATCH Commun. Math. Comput. Chem. **59** (2008) pp. 127–156.

[11] Liu J., *On harmonic index and diameter of graphs*, J. Appl. Math. Phys. **1** (2013) pp. 13–14.

[12] Liu J., *On the harmonic index of triangle-free graphs*, Appl. Math. **4** (2013) pp. 1204–1206.

[13] Randić M., *On characterization of molecular branching*, J. Am. Chem. Soc. **97** (1975) pp. 6609–6615.

[14] Tomescu I., Marinescu-Ghemeci R., Mihai G., *On dense graphs having minimum Randić index*, Romanian J. Inf. Sci. Tech. **12** (2009) pp. 455–465.

[15] Tomescu I., Kanwal S., *Ordering trees having small general sum-connectivity index*, MATCH Commun. Math. Comput. Chem. **69** (2013) pp. 535–548.

[16] Wu R., Tang Z., Deng H., *On the harmonic index and the girth of a graph*, Utilitas Math. **91** (2013) pp. 65–69.

[17] Xu X., *Relationships between harmonic index and other topological indices*, Appl. Math. Sci. **6** (2012) pp. 2013–2018.

[18] Yang L., Hua H., *The harmonic index of general graphs, nanocones and triangular benzenoid graphs*, Optoelectron. Adv. Mater. - Rapid Commun. **6** (2012) pp. 660–663.

[19] Zhong L., *The harmonic index for graphs*, Appl. Math. Lett. **25** (2012) pp. 561–566.

[20] Zhong L., *The harmonic index on unicyclic graphs*, Ars Combin. **104** (2012) pp. 261–269.

[21] Zhong L., Xu K., *The harmonic index for bicyclic graphs*, Utilitas Math. **90** (2013) pp. 23–32.

[22] Zhong L., Cui Q., *The harmonic index for unicyclic graphs with given girth*, preprint.

[23] Zhou B., Trinajstić N., *On general sum-connectivity index*, J. Math. Chem. **47** (2010) pp. 210–218.

[24] Zhu Y., Chang R., Wei X., *The harmonic index on bicyclic graphs*, Ars Combin. **110** (2013) pp. 97–104.

[25] Zhu Y., Chang R., *Minimum harmonic indices of trees and unicyclic graphs with given number of pendant vertices and diameter*, preprint.