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Online Algorithms for 3-Space Bounded 2-Dimensional Bin Packing and Square Packing

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Abstract. In 2-dimensional bin packing problem each item is a rectangle of side lengths not greater than 1. The items are packed online into square bins of size 1×1 and 90°-rotations are allowed. In t-space bounded model of online bin packing each item can be packed only into one of t active bins. If it is impossible to pack an item into any active bin, we close one of the current active bins and open a new active bin to pack that item. In this paper a 3.577-competitive 3-space bounded online packing algorithm is presented. Furthermore, an online algorithm for packing squares with the competitive ratio 2.8 is described.

1. Introduction

In online version of packing, items arrive over time, and when packing the current item, we have no information about the next items. The position of any packed item cannot be changed. In t-space bounded model, there are t active bins, and each item can be packed only into one of the active bins. If it is impossible to pack an item into any active bin, we close one of the current active bins and open a new active bin to pack that item. Once an active bin has been closed, it can never become active again.

Let S be a sequence of items, let $A(S)$ be the number of bins used by algorithm A and let $OPT(S)$ be the minimum possible number of bins used to pack items from S. The *asymptotic competitive ratio* for algorithm A is defined as:

$$R_A^\infty = \lim_{n\to\infty} \sup_S \{\frac{A(S)}{OPT(S)} \mid OPT(S) = n\}.$$

In 2-dimensional bin packing problem each item is a rectangle of side lengths not greater than 1. The items are packed into square bins of size 1×1 and 90°-rotations are allowed.

For d-dimensional hyperbox packing, Epstein and van Stee (see [8]) presented a $(\Pi_\infty)^d$-competitive space bounded algorithm, where $\Pi_\infty \approx 1.69103\ldots$ is the competitive ratio of the one-dimensional harmonic algorithm (see [13]). Algorithms with only one active bin were presented for the first time in [4] (see also [2]). A 1-space bounded 2-dimensional online packing strategy with the competitive ratio 5.06 and a 4.3-competitive 1-space bounded square packing method is given in [3]. The case with two active bins was also considered. A 4-competitive 2-space bounded 2-dimensional algorithm and a 3.8-competitive 2-space bounded square packing method is described in [12].

On the other hand, it is known that the asymptotic competitive ratio for any t-space bounded 2-dimensional packing algorithm is not smaller than $(\Pi_\infty)^2 \approx 2.85958\ldots$ for $t \geq 2$ (see [6]) and it is not smaller than 3.17 for $t = 1$ (see [3]). In the case of packing squares, a lower bound of 2.36343 (for $t \geq 2$) on the competitive ratio is given in [9] and a lower bound of 2.94 (for $t = 1$) is presented in [3].

In this paper we focus on the problem of online packing with 3 active bins. For packing squares, the authors use a modified version of a standard harmonic algorithm first described by Lee and Lee in [13]. Applying standard harmonic for packing into 2 or 3 bins is rather inefficient. Epstein and van Stee in [8] introduced special rules for packing the smallest items, and since it significantly lowered the competitive ratio compared to standard harmonic, it could still be improved in the case of a small number of bins.

We describe an online algorithm for 3-space bounded packing squares with competitive ratio 2.8. Furthermore, we present a 3.577-competitive 3-space bounded 2-dimensional online packing algorithm.

Bounds for online square packing algorithms are presented also in [4], [7] and [10]. Results concerning offline packing of rectangles into squares are discussed in [1], [5] and [11].

2. 3-space bounded square packing algorithm

Let $\mathcal{I}$ be a bin. We dissect it into squares.

For each non-negative integer k, let 2_k-*square* be a square of side length $1/(2 \cdot 2^k)$ and let 3_k-*square* be a square of side length $1/(3 \cdot 2^k)$. We can dissect any 2_k-square into four 2_{k+1}-squares. Furthermore, any 3_k-square can be dissected into four 3_{k+1}-squares.

We number all the four 2_0-squares of $\mathcal{I}$ by integers from 1 to 4 and all the nine 3_0-squares of $\mathcal{I}$ by integers from 1 to 9. Moreover, for each positive integer k we number all 3_k-squares and all 2_k-squares so that the numbers $4l-3$, $4l-2$, $4l-1$ and $4l$ are assigned to four squares obtained by dissection of the square with number l.

Let S be a sequence of square items $s_1, s_2, \ldots$. Denote by h_i the side length of s_i. We divide items into four types:

- an item s_i is 2^+ provided $h_i > 1/2$,
- an item s_i is 2^- provided $1/3 < h_i \le 1/2$,
- an item s_i is 2_k^- provided $1/(3 \cdot 2^k) < h_i \le 1/(2 \cdot 2^k),\ k \ge 1$,
- an item s_i is 3_k^- provided $1/(2 \cdot 2^{k+1}) < h_i \le 1/(3 \cdot 2^k),\ k \ge 0$.

Three active bins are open at any time of the packing process: first (called $\mathcal{B}^1$) for packing 2^+ and some 2^- items, second (called $\mathcal{B}^2$) for packing the rest of 2^- and 2_k^- items and third (called $\mathcal{B}^3$) for packing 3_k^- items. 2^+ and 2^- items are always packed into a corner of $\mathcal{B}^1$ bin. If $j < i$, then we say that s_j *precedes* s_i. Let $\xi \in \{2, 3\}$. A ξ_k-square is i-free, provided its interior has an empty intersection with every packed item preceding s_i.

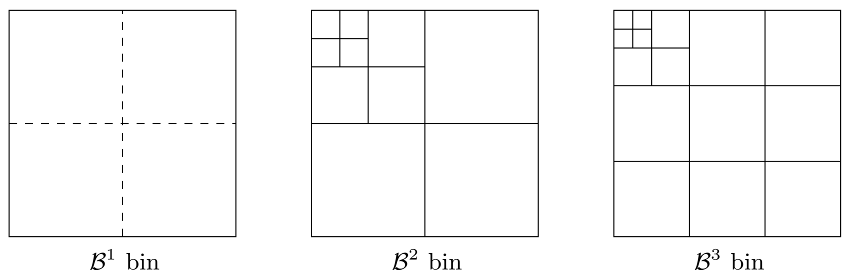

Fig. 1. three active bins.

In the standard 2-dimensional 3-space bounded harmonic algorithm all squares of side length greater than 1/2 are packed into $\mathcal{B}^1$, all squares of side length in $(1/3, 1/2]$ are packed into $\mathcal{B}^2$ and all smaller squares are packed into $\mathcal{B}^3$. Such algorithm has the competitive ratio 4.5. Epstein and van Stee's version of harmonic algorithm (see [8]) packs all 2^+ items into $\mathcal{B}^1$, all 2^- and 2_k^- items into $\mathcal{B}^2$ and 3_k^- items into $\mathcal{B}^3$. This algorithm has the competitive ratio equal to 3.25. The $P3S$ algorithm described below allows a 2^- item to be packed into $\mathcal{B}^1$ when it cannot be fitted into $\mathcal{B}^2$. Such modification improves the occupied area of $\mathcal{B}^2$ from 1/3 to 5/12 lowering the competitive ratio to 2.8.

Algorithm $P3S$: packing strategy for 3-space bounded packing of squares.

1. If s_i is 3_k^-, then we pack s_i into $\mathcal{B}^3$ in the i-free 3_k-square with the smallest possible number. If there is no i-free 3_k-square in $\mathcal{B}^3$, we close $\mathcal{B}^3$, open a new active $\mathcal{B}^3$ and pack s_i into the first 3_k-square in the new-created $\mathcal{B}^3$.

2. If s_i is 2_k^-, then we pack s_i into $\mathcal{B}^2$ in the i-free 2_k-square with the smallest possible number. If there is no i-free 2_k-square in $\mathcal{B}^2$, then we close this bin. Moreover:

(a) If $\mathcal{B}^1$ is either empty or a 2^+ item is in it, then a new $\mathcal{B}^2$ bin is opened.

(b) If $\mathcal{B}^1$ contains only 2^- items, then $\mathcal{B}^1$ is renamed to $\mathcal{B}^2$ and a new $\mathcal{B}^1$ bin is opened.

We pack s_i into the first 2_k-square of the new-created (or renamed) $\mathcal{B}^2$.

3. If s_i is 2^-, then we pack s_i into one of the i-free 2_0-squares of $\mathcal{B}^2$. If there is no i-free 2_0-square in $\mathcal{B}^2$, then:

   (a) If it is possible, then s_i is packed into a corner of $\mathcal{B}^1$. If after packing s_i, there are four items in $\mathcal{B}^1$, then $\mathcal{B}^1$ is closed and a new $\mathcal{B}^1$ bin is opened.

   (b) If s_i cannot be packed into $\mathcal{B}^1$ either, then $\mathcal{B}^1$ and $\mathcal{B}^2$ are closed, two new bins are opened and named $\mathcal{B}^1$ and $\mathcal{B}^2$. The item s_i is packed into a corner of $\mathcal{B}^2$.

4. If s_i is 2^+, then it is packed into a corner of $\mathcal{B}^1$. If this is impossible, then:

   (a) If $\mathcal{B}^1$ is occupied by another 2^+ item, then $\mathcal{B}^1$ is closed.

   (b) If there is no 2^+ item in $\mathcal{B}^1$, then $\mathcal{B}^2$ is closed and $\mathcal{B}^1$ is renamed to $\mathcal{B}^2$.

   A new $\mathcal{B}^1$ bin is opened (for a single packing) and s_i is packed into it. The bin is immediately closed and a new $\mathcal{B}^1$ bin is opened.

Lemma 1. *The total area of items from S packed into any closed $\mathcal{B}^3$ bin in $P3S$ strategy is not smaller than* $1/2$*.*

Proof. Squares packed into $\mathcal{B}^3$ satisfy $1/(2 \cdot 2^{k+1}) < h_i \leq 1/(3 \cdot 2^k)$. Each 3_k^- item is packed into a 3_k-square occupying at least

$$(3 \cdot 2^k)^2/(2 \cdot 2^{k+1})^2 = 9/16$$

of its area.

Let s_i be the first 3_k^- item that cannot be packed into $\mathcal{B}^3$. A 3_l-square is *empty* provided it is i-free and it is not contained in any i-free 3_q-square, for $q < l$. There is no empty 3_k-square and there could be at most 3 empty 3_n-squares for each $n > k$. The area of these empty squares is not greater than

$$\sum_{n>k} \frac{3}{(3 \cdot 2^n)^2} = \frac{1}{9} \cdot \frac{1}{2^{2k}} \leq \frac{1}{9}.$$

Consequently, the area of items packed into a closed $\mathcal{B}^3$ is not smaller than $(1 - 1/9) \cdot 9/16 = 1/2$. □

Lemma 2. *Let S be a sequence of items of finite total area and let m denote the number of 2^+ items in S. Furthermore, let m_i (for $i = 1, 2$) denote the number of closed $\mathcal{B}^i$ bins in $P3S$ strategy. The total area of items packed into m_1 closed $\mathcal{B}^1$ bins and into m_2 closed $\mathcal{B}^2$ bins is greater than* $\frac{1}{4}m + \frac{5}{12}(m_1 + m_2 - m)$*.*

Proof. There are five cases to close bins $\mathcal{B}^1$ and $\mathcal{B}^2$ (see the rules 2, 3a, 3b, 4a and 4b).

Case 1: $\mathcal{B}^2$ is closed because there is no i-free 2_k-square to pack a 2_k^- item s_i (see the rule 2). Calculations analogous to those in the proof of the Lemma 1 show that the area of all empty squares in this case is at most

$$\frac{1}{4} \cdot \frac{1}{2^{2k}}.$$

Since s_i is a 2_k^- item for $k \geq 1$, the area of empty squares is at most $1/16$. Each 2_k^- item is packed into a 2_k-square occupying at least

$$(2 \cdot 2^k)^2/(3 \cdot 2^k)^2 = 4/9$$

of its area. Finally, the total occupied area is not smaller than

$$(1 - 1/16) \cdot 4/9 = 5/12.$$

From now on, we will assume that the area of each 2^- item packed into a 2_0-square equals $1/9$. Excess area gives us additional profit, which is discussed in details in Case 4.

Case 2: $\mathcal{B}^1$ is closed because four items are packed in it (see the rule 3a). There are 2^+ or 2^- items in $\mathcal{B}^1$, thus the total area of items is at least $4 \cdot (1/3)^2 > 5/12$.

Case 3: $\mathcal{B}^1$ is closed because a 2^+ item s_i cannot be packed into it and another 2^+ item s_j was packed into it (see the rule 4a). Obviously, $h_j + h_i > 1$ and $h_i^2 + h_j^2 > 1/2$. This implies that the average occupied area in each $\mathcal{B}^1$ bin is at least $1/4$.

Case 4: $\mathcal{B}^1$ is closed because a 2^- item s_i cannot be packed in either $\mathcal{B}^1$ or $\mathcal{B}^2$ (see the rule 3b). In this case we close two bins: $\mathcal{B}^1$ and $\mathcal{B}^2$. Using reasoning presented in Case 1 we conclude that the total area of items packed into $\mathcal{B}^2$ is greater than $(1 - 1/4) \cdot 4/9 = 1/3$. As in Case 3, the sum of areas of items packed into $\mathcal{B}^1$ plus the area of s_i is greater than $1/2$.

We pack s_i into a quarter of the new-created $\mathcal{B}^2$ bin. As was mentioned at the end of Case 1, in our calculations we assume that the area of s_i equals $1/9$. In fact, the total area of items packed into two closed bins plus the area of s_i is greater than

$$\frac{1}{3} + \frac{1}{2} > \frac{5}{12} + \frac{1}{4} + \frac{1}{9}.$$

We can assume that the average occupied area in the closed $\mathcal{B}^2$ bin is $5/12$ and in $\mathcal{B}^1$ is $1/4$.

Case 5: $\mathcal{B}^1$ is closed because a 2^+ item s_i cannot be packed into it and no 2^+ item s_j was packed into it before (see the rule 4b). This case is similar to Case 4. The total area of items packed into two closed bins $\mathcal{B}^1$ and $\mathcal{B}^2$ plus the area of s_i (this item is packed into a quarter of the "old" $\mathcal{B}^1$ renamed to $\mathcal{B}^2$) is greater than $\frac{1}{3} + \frac{1}{2} > \frac{5}{12} + \frac{1}{4} + \frac{1}{9}$.

□

Theorem 1. *The asymptotic competitive ratio of the strategy P3S is not greater than* 2.8*.*

Proof. Let S be a sequence of items of total area a and let m denote the number of 2^+ items in S. Obviously, $OPT(S) \geq a$ provided $a > m$ and $OPT(S) \geq m$ provided $a \leq m$.

Let m_i (for $i = 1, 2, 3$) denote the number of closed $\mathcal{B}^i$ bins in the $P3S$ strategy. By Lemma 1 and Lemma 2 we know that

$$\frac{1}{4}m + \frac{5}{12}(m_1 + m_2 - m) + \frac{1}{2}m_3 \leq a.$$

Consequently,

$$m_1 + m_2 + m_3 \leq m + \frac{12}{5}(a - \frac{1}{4}m).$$

Denote by $P3S(S)$ the number of bins used for packing items from S by $P3S$ strategy. At any time of the packing process there are three open bins.

If $a > m$, then

$$\begin{aligned}\frac{P3S(S)}{OPT(S)} &\leq \frac{m + \frac{12}{5}(a - \frac{1}{4}m) + 3}{a} = \frac{\frac{12}{5}a + \frac{2}{5}m + 3}{a} \\ &< \frac{\frac{12}{5}a + \frac{2}{5}a + 3}{a} = 2.8 + \frac{3}{a}.\end{aligned}$$

If $a \leq m$, then

$$\frac{P3S(S)}{OPT(S)} \leq \frac{\frac{12}{5}a + \frac{2}{5}m + 3}{m} \leq \frac{\frac{12}{5}m + \frac{2}{5}m + 3}{m} = 2.8 + \frac{3}{m}.$$

Consequently, the asymptotic competitive ratio for $P3S$ is not greater than 2.8. □

3. k-containers

Let $\mathcal{I}$ be a bin. For each positive integer k, let k-container be a rectangle of sides of length $1/2^k$ and $4/2^k$.

Obviously, $\mathcal{I}$ can be partitioned into 4^{k-1} k-containers for $k \geq 2$. We number all 2-containers of $\mathcal{I}$ by integers from 1 to 4 as in Fig. 2.

For each integer $k \geq 3$, we number also all k-containers of $\mathcal{I}$ so that the numbers $4l - 3$, $4l - 2$, $4l - 1$ and $4l$ are assigned to four k-containers obtained by dissection of the $(k-1)$-container with number l as in Fig. 3.

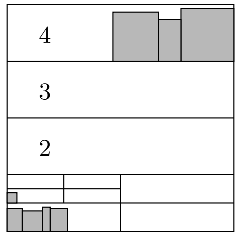

Fig. 2. 2-containers.

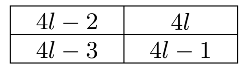

Fig. 3. k-containers.

4. 3-space bounded packing algorithm

Let S be a sequence of items $s_1, s_2, \ldots$. Denote by h_i the height and by w_i the width of s_i. We can assume that $h_i \geq w_i$.

The idea of packing items is standard, each bin can be used to pack a particular type of items. If $w_i > 1/2$, then s_i is *large*. If $h_i \geq 1/2$ and if $w_i \leq 1/2$, then s_i is *very big*. If $1/4 < h_i < 1/2$, then s_i is *big* and if $h_i \leq 1/4$, then s_i is *small*. A small item s_i is *of type* k provided $2^{-k-1} < h_i \leq 2^{-k}$.

Three active bins are open at any time in the packing process: one s-bin for packing small items and two b-bins ($\mathcal{B}_1$ and $\mathcal{B}_2$) for packing large, big and very big items. Small items of type k will be packed into k-containers. Each large or very big item will be placed in left-right order either along the bottom or along the top of a b-bin (as in $P2S1$ strategy described in [12]). Each big item will be packed in right-left order either along the bottom or along the top of a b-bin (as in $P2S1$ strategy presented in [12]).

If $j < i$, then we say that s_j *precedes* s_i.

For $m \in \{1, 2\}$ denote by t_i^m the sum of the widths of big items preceding s_i that have been packed into $\mathcal{B}_m$ along the top. Furthermore, denote by b_i^m the sum of the widths of big items preceding s_i that have been packed into $\mathcal{B}_m$ along the bottom.

If an item of type $n \geq 3$ preceding s_i has been packed into an n-container R, then all p-containers, for $p > n$, contained in R are i-*closed*. Moreover, all q-containers, where $2 < q < n$, that contain R are i-*closed*. If a packed item of type 2 preceding s_i has a non-empty intersection with the interior of a left or right half of a 2-container, then all p-containers, for $p \geq 3$, contained in this half are i-*closed*. For $k \geq 3$ all k-containers which are not i-closed are i-*open*. For example, assume that items $s_1, \ldots, s_{i-1}$ have been packed as in Fig. 2 ($i = 9$ here). Then there are eleven i-open 3-containers (3-containers of numbers 2, 13, 14, 15, 16 are i-closed).

Algorithm $P3$: packing strategy for 2-dimensional 3-space bounded bin packing.

(r_1) If s_i is a small item of type 2, then it is packed into the active s-bin. We find the 2-container with the greatest number into which s_i can be packed. We pack s_i

into this container as far to the right as it is possible. If there is no 2-container into which s_i can be packed we close this bin, open a new active s-bin and pack s_i into the fourth 2-container in the newly opened active s-bin as far to the right as it is possible.

(r_2) If s_i is a small item of type k for $k \geq 3$, then it is packed into the active s-bin. We find the i-open k-container with the smallest number into which s_i can be packed. We pack s_i into this container as far to the left as it is possible. We say that this container is *used for the packing*. If there is no i-open k-container into which s_i can be packed we close this bin, open a new active s-bin and pack s_i into the first k-container in the newly opened active s-bin as far to the left as it is possible. We say that this container is *used for the packing*.

(r_3) If s_i is big, then it is packed into $\mathcal{B}_1$ either along the bottom of this b-bin (provided $t_i^1 > b_i^1$) or along the top of this bin (provided $t_i^1 \leq b_i^1$) as far to the right as it is possible. If it is unfeasible, then we pack s_1 into $\mathcal{B}_2$ either along the bottom of this b-bin (provided $t_i^2 > b_i^2$) or along the top of this bin (provided $t_i^2 \leq b_i^2$) as far to the right as it is possible. If it is unfeasible, we close the b-bin that is more packed (i.e., the total area of items packed in this bin is not smaller than the total area of items packed in the other), open a new active b-bin and pack s_i in the right-up corner. Now the unclosed b-bin becomes $\mathcal{B}_1$ and the newly opened b-bin is $\mathcal{B}_2$.

(r_4) If s_i is very big, then it is packed into $\mathcal{B}_1$ either along the bottom of this bin (provided $t_i^1 \geq b_i^1$) or along the top of this bin (provided $t_i^1 < b_i^1$) as far to the left as it is possible. If it is unfeasible, then we pack s_i into $\mathcal{B}_2$ either along the bottom of this bin (provided $t_i^2 \geq b_i^2$) or along the top of this bin (provided $t_i^2 < b_i^2$) as far to the left as it is possible. If it is unfeasible, we close the active b-bin that is more packed, open a new active b-bin and pack s_i in the left-down corner. Now the unclosed b-bin becomes $\mathcal{B}_1$ and the newly opened b-bin is $\mathcal{B}_2$.

(r_5) If s_i is large, then it is packed into $\mathcal{B}_1$ either along the bottom of this bin (provided $t_i^1 \geq b_i^1$) or along the top of this bin (provided $t_i^1 < b_i^1$) as far to the left as it is possible. If it is unfeasible, then we close $\mathcal{B}_1$ and open a new active b-bin for a single packing. We pack s_i into this bin, close this bin and open a new b-bin. Now the newly opened bin is $\mathcal{B}_2$ and the unclosed b-bin becomes $\mathcal{B}_1$.

Lemma 3. *Assume that there is no large item in a sequence S. The total area of items from S packed into any closed b-bin in $P3$ strategy is not smaller than* $(3-\sqrt{3})/4 \approx 0.317$.

Proof. Put $\varrho = (3-\sqrt{3})/4 \approx 0.317$ and $\lambda = (\sqrt{3}-1)/2 \approx 0.366$.

Let s_z be the first big or very big item that cannot be packed in either $\mathcal{B}_1$ or $\mathcal{B}_2$. One of these two bins will be closed. Denote by Σ_1 the total area of items packed into $\mathcal{B}_1$ and by Σ_2 the total area of items packed into $\mathcal{B}_2$. We show that either $\Sigma_1 \geq \varrho$ or $\Sigma_2 \geq \varrho$, *i.e.*, that the occupied area in any closed b-bin is not smaller than ϱ.

It is impossible that a b-bin transforms from $\mathcal{B}_1$ into $\mathcal{B}_2$. Thus for each bin $\mathcal{B}_2$ there is only one corresponding bin $\mathcal{B}_1$. This implies that each item packed into $\mathcal{B}_2$ could not be packed into $\mathcal{B}_1$.

Since there is no large item in the sequence, at least two items were packed into $\mathcal{B}_2$.

Assume that a very big item was packed into $\mathcal{B}_2$. If $t_z^1 = b_z^1$, then all very big items packed into $\mathcal{B}_2$ are *low*. Otherwise, by the *leftmost* packed big item we mean the big item packed in $\mathcal{B}_1$ with the greatest distance between its left side and the right side of $\mathcal{B}_1$. A very big item packed into $\mathcal{B}_2$ is *tall* [is *low*] provided $h + H > 1$ [provided $h + H \le 1$, respectively], where H is the height of this very big item and h is the height of the leftmost packed big item.

Denote by w the sum of widths of very big items packed into $\mathcal{B}_1$ and put $u = |t_z^1 - b_z^1|$ and $v = 1 - w - \max(t_z^1, b_z^1)$ (see Fig. 4).

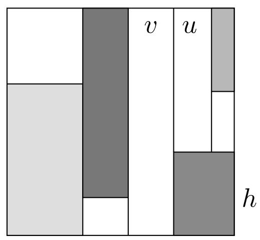

Fig. 4. Items in $\mathcal{B}_1$.

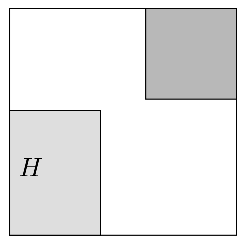

Fig. 5. $\mathcal{B}_2 :\ h + H \le 1$.

Case 1: there is no tall item in B_2. If $v + u \le \lambda$, then

$$\Sigma_1 \ge \frac{1}{2}(1 - v - u) \ge \frac{1}{2}(1 - \lambda) = \varrho.$$

If $v + u > \lambda$, then either

$$\Sigma_2 > \frac{1}{2}(v + u) + (v + u)^2 > \frac{1}{2}\lambda + \lambda^2 = \varrho$$

(see Fig. 5) or

$$\Sigma_2 > 2 \cdot \frac{1}{2}(v + u) > \lambda > \varrho$$

(see Fig. 6).

Case 2: there is at least one tall item in $\mathcal{B}_2$.

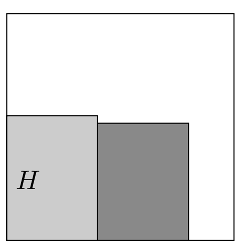

Fig. 6. $h + H \le 1$.

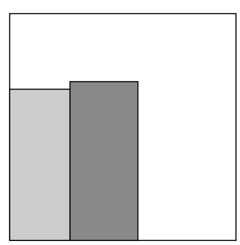

Fig. 7. Tall items.

Subcase 2a: there are only tall items in $\mathcal{B}_2$ (see Fig. 7, the height of each item in $\mathcal{B}_2$ is greater than $1-h$). Since there is no large item in $\mathcal{B}_2$, it follows that $v < \frac{1}{2}$. Obviously, $\Sigma_2 > \frac{1}{2}(1-h)$ as well as $\Sigma_2 > 2v(1-h)$. Consequently, if $\Sigma_2 < \varrho$, then $h > 1-2\varrho = \lambda$ and $v < \frac{\varrho}{2(1-h)}$. By $u \le h < 1/2$ and $h > \lambda$ we have

$$\begin{aligned}\Sigma_1 &\ge \frac{1}{2}(1-v-u)+uh \ge \frac{1}{2}(1-v-h)+h^2 \\ &> \frac{1}{2}-\frac{\varrho}{4(1-h)}-\frac{1}{2}h+h^2 \\ &> \frac{1}{2}-\frac{\varrho}{4(1-\lambda)}-\frac{1}{2}\lambda+\lambda^2 > \varrho.\end{aligned}$$

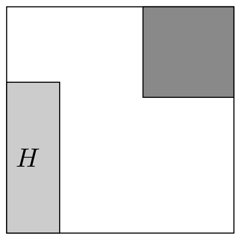

Fig. 8. $h+H>1$.

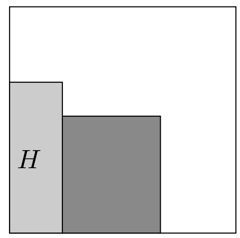

Fig. 9. $h+H>1$.

Subcase 2b: there is a big item in $\mathcal{B}_2$ (see Fig. 8). If $\frac{1}{2}(1-v-u)+uh \ge \varrho$, then $\Sigma_1 \ge \varrho$. Otherwise,

$$v > 1-u+2uh-2\varrho$$

and

$$\begin{aligned}\Sigma_2 &> v(1-h)+(v+u)^2 \\ &> (1-u+2uh-2\varrho)(1-h)+(1+2uh-2\varrho)^2 \\ &= 4u^2h^2-2uh^2+7uh-8uh\varrho-u-h+2h\varrho+2-6\varrho+4\varrho^2.\end{aligned}$$

For $\frac{1}{4} \le h \le \frac{1}{2}$ and $0 \le u \le h$, this value is minimal at $u=0$ and $h=\frac{1}{2}$. Consequently,

$$\Sigma_2 > \frac{3}{2}-5\varrho+4\varrho^2 = \varrho.$$

Subcase 2c: there is a low item in $\mathcal{B}_2$ (see Fig. 9). If $v(1-h)+\frac{1}{2}(v+u) \ge \varrho$, then $\Sigma_2 \ge \varrho$. Otherwise,

$$v < \frac{2\varrho-u}{3-2h}.$$

Hence

$$\begin{aligned}\Sigma_1 &\ge \frac{1}{2}(1-v-u)+uh > \frac{1}{2}\Big(1-\frac{2\varrho-u}{3-2h}-u\Big)+uh \\ &= \frac{1}{2}-\frac{\varrho}{3-2h}+u\cdot\frac{-2h^2+4h-1}{3-2h}.\end{aligned}$$

Obviously, $0 \le u \le h$ and $\frac{1}{4} \le h \le \frac{1}{2}$.

If $h \ge 1 - \sqrt{2}/2$, then $-2h^2 + 4h - 1 \ge 0$ and

$$\Sigma_1 > \frac{1}{2} - \frac{\varrho}{3-2h} > \varrho.$$

If $h < 1 - \sqrt{2}/2$, then

$$\begin{aligned} \Sigma_1 &> \frac{1}{2} - \frac{\varrho}{3-2h} + h \cdot \frac{-2h^2+4h-1}{3-2h} \\ &> \frac{1}{2} - \frac{\varrho}{3-2/4} + \frac{1}{4} \cdot \frac{-2/16+4/4-1}{3-2/4} > \varrho \end{aligned}$$

□

Remark. The bound of $\varrho = (3-\sqrt{3})/4 \approx 0.317$ in Lemma 1 cannot be lessened. Let $x_1 = x_2 = \left(\frac{1}{2} - \frac{1}{2}\lambda\right) \times \left(\frac{1}{2} + \epsilon\right)$, $x_3 = \epsilon \times 1$, $x_4 = \lambda \times \left(\frac{1}{2} + \epsilon\right)$, $x_5 = \lambda \times \lambda$ and $x_6 = \left(\frac{1}{2} - \epsilon\right) \times 1$. For small ϵ, items x_1, x_2 and x_3 are packed by $P3$ in $\mathcal{B}_1$, items x_4 and x_5 are packed by $P3$ in $\mathcal{B}_2$ and x_6 cannot be packed in either $\mathcal{B}_1$ or $\mathcal{B}_2$. Obviously, $\Sigma_1 = \varrho + 2\epsilon - \lambda\epsilon$ and $\Sigma_2 = \varrho + \lambda\epsilon$.

Lemma 4. *The total area of items packed into any closed s-bin in P3 strategy is greater than* $(3-\sqrt{3})/4 \approx 0.317$.

Proof. Let $\mathcal{A}$ be a closed s-bin and let s_z be the first small item that cannot be packed into $\mathcal{A}$. For each integer $k \ge 2$ such that a small item of type k has been packed into $\mathcal{A}$ denote by $R_1^k, \ldots, R_{l_k}^k$ all k-containers of $\mathcal{A}$ used for the packing. We say that $R_1^k, \ldots, R_{l_k-1}^k$ are *full* and that R_{l_k} is *partially packed*. Moreover, let s_{j_i} be the first item of type k packed into R_i, for $i = 1, \ldots, l_k$ (see Fig. 10 and Fig. 11).

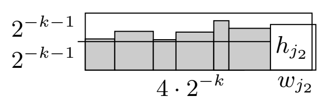

Fig. 10. R_1.

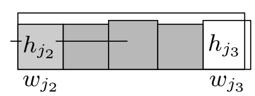

Fig. 11. R_2.

The total area of items of type k packed into $R_1 \cup \ldots \cup R_{l_k}$ is greater than

$$\sum_{i=2}^{l} \left[2^{-k-1}\left(4 \cdot 2^{-k} - w_{j_i}\right) + w_{j_i}\left(h_{j_i} - 2^{-k-1}\right)\right].$$

This value is minimal at $w_{j_2} = h_{j_2} = \ldots = w_{j_l} = h_{j_l} = 2^{-k-1}$. Thus the total area of packed items of type k is greater than $\frac{7}{16}$ times the total area of full k-containers.

The sum of areas of partially packed k-containers, for $k \ge 3$, is smaller than

$$\frac{1}{8} \cdot \frac{1}{2} + \frac{1}{16} \cdot \frac{1}{4} + \ldots = \frac{1}{12}.$$

A k-container, for $k \geq 3$, is *empty* provided it is z-open, it is not used for the packing and it is not contained in any empty q-container for $3 \leq q < k$.

If no item of type 2 has been packed into $\mathcal{A}$, then we put $Q = \emptyset$. Otherwise, denote by P the 2-container with the smallest number into which an item of type 2 has been packed. If the sum of the widths of items of type 2 packed into P is smaller than 1/2, then let Q be the right half of P (see Fig. 12). Otherwise, let Q be the left half of P (see Fig. 13). Obviously, if Q is the left half of P, then the occupied area in the right half of P is greater than $\frac{7}{16}$ times the area of $P \setminus Q$.

Case 1: s_z is of type n, where $n \geq 3$ (see Fig. 12). There is no empty 3-container in $\mathcal{A}$. Moreover, there could be at most three empty k-containers for any $k \geq 4$. Consequently, the sum of areas of empty containers is smaller than

$$3 \cdot \frac{1}{16} \cdot \frac{1}{4} + 3 \cdot \frac{1}{32} \cdot \frac{1}{8} + \ldots = \frac{1}{16}.$$

Thus the total area of packed items is greater than

$$\frac{7}{16}\left[1 - \frac{1}{12} - \frac{1}{16} - area(Q)\right] \geq \frac{7}{16}\left[1 - \frac{1}{12} - \frac{1}{16} - \frac{1}{8}\right] = \frac{245}{768} \approx 0.319.$$

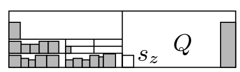

Fig. 12. Items in P.

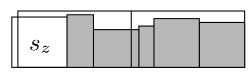

Fig. 13. s_z is of type 2.

Case 2: s_z is of type 2 (see Fig. 13). If an item of type j, where $j \geq 3$, has been packed into Q, then as in Case 1 we obtain that the total area of packed items is greater than

$$\frac{7}{16}\left[1 - \frac{1}{12} - \frac{1}{16} - area(Q)\right] \geq \frac{7}{16}\left[1 - \frac{1}{12} - \frac{1}{16} - \frac{1}{8}\right] \approx 0.319.$$

If only items of type 2 have been packed into Q, then the occupied area in Q is greater than 1/4 times the area of Q (s_z cannot be packed into Q). Obviously, there is at most one empty 3-container (s_z cannot be packed into $\mathcal{A}$). Moreover, there could be at most three empty k-containers for any $k \geq 4$. This implies that the sum of areas of empty containers is smaller than

$$\frac{1}{8} \cdot \frac{1}{2} + 3 \cdot \frac{1}{16} \cdot \frac{1}{4} + 3 \cdot \frac{1}{32} \cdot \frac{1}{8} + \ldots = \frac{1}{8}.$$

Consequently, the occupied area in $\mathcal{A}$ is greater than

$$\frac{7}{16}\left[1 - \frac{1}{12} - \frac{1}{8} - area(Q)\right] + \frac{1}{4} \cdot area(Q) \geq \frac{31}{96} \approx 0.323.$$

$\square$

Theorem 2. *The asymptotic competitive ratio of the strategy $P3$ is not smaller than $3+\sqrt{3}/3 \approx 3.577$.*

Proof. Let S be a sequence of items of total area a and let m denote the number of large items in S. Obviously, $OPT(S) \geq a$ provided $a > m$ and $OPT(S) \geq m$ provided $a \leq m$.

Large, big and very big items are packed as in $P2S1$ strategy described in [12]. In this method, on average, at least 1/4 of each bin is occupied. By the proof of Theorem 1 of [12] we deduce that m large items (possibly with some big and very big items) can be packed by $P3$ into m_0 bins, where $m \leq m_0 \leq 2m$, so that the total area of packed items is greater than $m_0/4$. By Lemmas 1 and 2 we know that the remaining items (of total area smaller than $a - m_0/4$) can be packed into $\lceil \frac{4}{3-\sqrt{3}}(a - \frac{1}{4}m_0)\rceil$ bins.

This implies that items from S can be packed into $m_0 + \lceil \frac{4}{3-\sqrt{3}}(a - \frac{1}{4}m_0)\rceil$ bins. If $a > m$, then

$$\begin{aligned}
\frac{P3(S)}{OPT(S)} &\leq \frac{m_0 + \lceil \frac{2}{3}(3+\sqrt{3})(a - \frac{1}{4}m_0)\rceil}{a} \\
&\leq \frac{2m + \lceil \frac{2}{3}(3+\sqrt{3})(a - \frac{1}{4}\cdot 2m)\rceil}{a} \\
&< \frac{2a + \frac{2}{3}(3+\sqrt{3})(a - \frac{1}{2}a) + 1}{a} \\
&= 3 + \frac{\sqrt{3}}{3} + \frac{1}{a}.
\end{aligned}$$

If $a \leq m$, then

$$\begin{aligned}
\frac{P3(S)}{OPT(S)} &\leq \frac{2m + \lceil \frac{2}{3}(3+\sqrt{3})(a - \frac{1}{2}m)\rceil}{m} \\
&< \frac{2m + \frac{2}{3}(3+\sqrt{3})(m - \frac{1}{2}m) + 1}{m} \\
&= 3 + \frac{\sqrt{3}}{3} + \frac{1}{m}.
\end{aligned}$$

Consequently, the asymptotic competitive ratio for $P3$ is not greater than $3+\sqrt{3}/3$. □

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