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Approximation of Köthe-Bochner Spaces $L_\rho(X)$ in case X has a Schauder Basis

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Abstract. We consider Köthe-Bochner spaces $L_\rho(X)$ (X Banach space, ρ function norm) in case X has a Schauder basis. The canonical finite dimensional subspaces Y of X generate the Köthe-Bochner spaces $L_\rho(Y)$, which approximate $L_\rho(X)$. Numerical computations in case X is a Lorentz sequence space are performed. We present a general computation and a special computation for a particular case, comparing the procedures and noticing that the special computation is more economic.

Key-words: Köthe spaces, Köthe-Bochner spaces, Schauder basis, Lorentz sequence spaces, measurable vector functions.

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1. Introduction

The history of Köthe spaces begins with the famous paper [3], in which G. Köthe and O. Toeplitz introduced the ancestors of Köthe spaces, the so called "gestufte Räume" (which were special sequence spaces). Subsequently, G. Köthe proved new properties concerning these sequence spaces. Whitin the more general framework of measure spaces, the Nederland school (especially A. C. Zaanen and W. A. J. Luxemburg) developped the theory and proposed the name "Köthe spaces" for the more general spaces of measurable functions studied by them. The Köthe spaces generalise the Orlicz, Lorentz and Marcinkiewicz spaces.

Increasing generality, the Köthe-Bochner spaces appeared: they are Köthe spaces of measurable functions taking values in Banach spaces X. These Köthe-Bochner spaces constitute the subject of this paper. More precisely, we consider the case when the Banach space X possesses a Schauder basis. Within this framework, we can study the approximation of the elements of a given Köthe-Bochner space with elements in the canonical subspaces.

Concrete computations and comparisons concerning the speeds of the approximation processes are performed at the end of this paper, in the special case of functions having their values in Lorentz sequence spaces.

The monographs used as theoretical basis are [1], [4] and [6]. For Functional Analysis, see [2]. For supplementary results, see also [5].

2. Preliminary Facts

Throughout the paper K will be the scalar field (either $K = \mathbb{R}$ or $K = \mathbb{C}$),$\mathbb{N} = \{1, 2, 3, ...\}$ will be the set of natural numbers, $\mathbb{R}_+ = [0, \infty)$ and $\overline{\mathbb{R}}_+ = [0, \infty) \cup \{\infty\}$.

For any nonempty set T, any vector space X, any $x \in X$ and any $f : T \to K$, we can define $fx : T \to X$ via $(fx)(t) = f(t)x$. For any $A \subset T$, the function $\varphi_A : T \to \mathbb{R}_+$, acting via $\varphi_A(t) = 0$, if $t \notin A$ and $\varphi_A(t) = 1$ if $t \in A$ is the characteristic (indicator) function of A. Write $\mathcal{P}(T) = \{A \mid A \subset T\}$.

For any Banach space X, we shall denote by X' the dual of X.

From now on, we assume that X is a Banach space having a Schauder basis $(e_n)_{n\geq 1}$. The corresponding biorthogonal functionals are $(x'_i)_{i\geq 1} \subset X'$. For any n, $P_n : X \to X$ is the canonical projection defined via $P_n(\sum_{i=1}^{\infty} \alpha_i e_i) = \sum_{i=1}^{n} \alpha_i e_i$. One knows that $(P_n)_{n\geq 1}$ is a bounded sequence, i.e. $\sup_n \|P_n\|_o = M < \infty$, where $\|P_n\|_o$ is the usual operator norm. We call M the constant of the Schauder basis. In case $M = 1$, we say that the Schauder base is monotone. *For any function* $f : T \to X$, *we define the functions* $f_n : T \to X$, $f_n(t) = (P_n \circ f)(t)$, which will be very important in the sequel.

If X is a Banach space consisting of numerical sequences (i.e. an element of X has the form $x = (x_n)_{n\geq 1}$ with $x_n \in K$), in many cases one has:
a) $e_n \in X$ for any n, where $e_n = (0, 0, ..., 0, 1, 0, ...)$, with 1 on the $n - th$ place;
b) $(e_n)_{n\geq 1}$ is a Schauder basis for X.

In this case, we shall say that $(e_n)_n$ is *the canonical Schauder basis for X*.

This happens if $X = l^p$, $1 \leq p < \infty$. In this case, one can see that $(e_n)_n$ is a monotone Schauder basis.

The remaining part of these preliminaries is dedicated to Köthe and Köthe-Bochner spaces.

Let $(T, \mathcal{T}, \mu)$ be a measure space ($\phi \neq T$, $\mathcal{T} \subset \mathcal{P}(T)$, $\mathcal{T}\sigma$-algebra, $\mu : \mathcal{T} \longrightarrow \overline{\mathbb{R}}_+$ measure, μ σ-finite and complete). Let $(X, \| \|)$ be a Banach space.

Write $M_X(\mu) = \{f : T \longrightarrow X \mid f \text{ is } \mu\text{-measurable}\}$.

Hence, for any $f \in M_X(\mu)$, there exists a sequence $(f_n)_n \subset \mathcal{E}_X(\mu)$ such that $f_n \underset{n}{\to} f$ pointwise $\mu - a.e.$, where $\mathcal{E}_X(\mu)$ is the space of μ-simple functions with values in X.

Write $M_+(\mu) = \{u : T \to \overline{\mathbb{R}}_+ \mid u \text{ is } \mu\text{-measurable}\}$.

A function norm is a function $\rho : M_+(\mu) \to \overline{\mathbb{R}}_+$ with the properties:

i) $\rho(u) = 0 \Leftrightarrow u = 0 \ \mu - a.e.$;

ii) $\rho(u) \le \rho(v)$, whenever $u \le v$;

iii) $\rho(u + v) \le \rho(u) + \rho(v)$;

iv) $\rho(\alpha u) = \alpha\rho(u)$, with the convention $0 \cdot \infty = 0$, for any $u, v \in M_+(\mu)$ and any $\alpha \in \mathbb{R}_+$.

Example 2.1. For $1 \le p < \infty$,

$$\rho(u) = \|u\|_p = (\textstyle\int u^p d\mu)^{\frac{1}{p}}.$$

We say that ρ *has the Riesz-Fischer property* (and write ρ R-F) if

$$\rho(\sum_{n=1}^{\infty} u_n) \le \sum_{n=1}^{\infty} \rho(u_n),$$

for any sequence $(u_n)_n$ in $M_+(\mu)$.

We say that ρ *is of absolutely continuous type* (and write ρ *a.c.*) if

$$\rho(u_n) \downarrow 0,$$

for any decreasing sequence $(u_n)_n$ in $M_+(\mu)$ such that $\rho(u_1) < \infty$ and $u_n \downarrow 0$ (pointwise).

For any $1 \le p < \infty$, $\| \ \|_p$ R-F and $\| \ \|_p$ *a.c.*

For any $f : T \to X$, we shall write $|f|$ to designate the function $|f| : T \to \mathbb{R}_+$, acting via

$$|f|(t) = \|f(t)\|.$$

If $f \in M_X(\mu)$, then $|f| \in M_+(\mu)$ and one can compute

$$\rho|f| = \rho(|f|) \le \infty.$$

Define $\mathcal{L}_\rho(X) = \{f \in M_X(\mu) \mid \rho|f| < \infty\}$.

Definition 2.2. The spaces $\mathcal{L}_\rho(X)$ are called *(seminormed) Köthe-Bochner spaces.*

Then $\mathcal{L}_\rho(X)$ is seminormed with the seminorm

$$f \to \rho|f|.$$

The null space of this seminorm is

$$\mathcal{N}_\rho(X) = \{f \in \mathcal{L}_\rho(X) \mid \rho|f| = 0\}.$$

One can see that

$$\mathcal{N}_\rho(X) = \{f \in M_X(\mu) \mid f = 0 \ \mu - a.e.\}.$$

The associated normed space is

$$L_\rho(X) = \mathcal{L}_\rho(X)/\mathcal{N}_\rho(X),$$

normed with the norm

$$\tilde{f} \to \rho|f| = \left\|\tilde{f}\right\|,$$

for any $f \in \tilde{f} \in L_\rho(X)$.

Definition 2.3. The spaces $L_\rho(X)$ are called *Köthe-Bochner spaces.*

Theorem 2.4. *If ρ R-F, $L_\rho(X)$ is a Banach space.*

Example 2.5. If $\rho = \| \ \|_p$, $\mathcal{L}_\rho(X) = \mathcal{L}^p(X)$, $L_\rho(X) = L^p(X)$ (the Lebesgue spaces).

In case $X = K$, we write $M(\mu)$, $\mathcal{L}_\rho$, L_ρ instead of $M_X(\mu)$, $\mathcal{L}_\rho(X)$ and $L_\rho(X)$ respectively. If $X = \mathbb{R}$, L_ρ is a normed lattice.

3. Results

Theorem 3.1. *Any $f \in \mathcal{L}_\rho(X)$ can be uniquely written as follows (pointwise convergence):*

$$f = \sum_{i=1}^{\infty} a_i e_i,$$

where $a_i \in \mathcal{L}_\rho$, for any i.

Proof. Fixing $t \in T$ and x'_n (where $(x'_i)_i$ are the biorthogonal functionals of the basis $(e_n)_n$) we get

$$a_n(t) = x'_n(f(t)),$$

hence

$$f(t) = \sum_{n=1}^{\infty} a_n(t) e_n.$$

We got the functions $(a_n)_n$, $a_n : T \to K$, which are $\mu-$measurable because $a_n = x'_n \circ f$. We infer that all a_n are in $\mathcal{L}_\rho$. To this end, let $F_n = (P_n - P_{n-1}) \circ f$ for $n \geq 2$ and notice that, for any i and any t, one has

$$\|F_i(t)\| \leq 2M \, \|f(t)\|,$$

where M is the constant of the Schauder basis $(e_n)_n$. Because $F_i(t) = a_i(t)e_i$ it follows that

$$|a_i| \le \frac{2M}{\|e_i\|}|f|,$$

completing the proof of the fact that $a_i \in \mathcal{L}_\rho$. □

The main result is the following:

Theorem 3.2. (Approximation theorem) *Assume ρ a.c. Then, for any $f \in \mathcal{L}_\rho(X)$, one has $f_n \underset{n}{\rightarrow} f$ in $\mathcal{L}_\rho(X)$.*

Proof. As previously, $f = \sum_{i=1}^{\infty} a_i e_i$, with $a_i \in \mathcal{L}_\rho$. For any $t \in T$,

$$\|f_n(t)\| = \|P_n(f(t))\| \le M\,\|f(t)\|\,, \tag{1}$$

hence $|f_n| \le M|f|$, for any $n \in \mathbb{N}$.

For any , let us define $u_n : T \to \overline{\mathbb{R}}_+$, via

$$u_n(t) = \sup_m \|f_{n+m}(t) - f(t)\|\,.$$

Hence, u_n are μ-measurable functions and the sequence $(u_n)_n$ is decreasing. Due to the inequality $\|f_{n+m}(t) - f(t)\| \le \|f_{n+m}(t)\| + \|f(t)\|$, and (1) we get $u_n \le (M+1)|f|$, hence $\rho(u_n) < \infty$, for any n.

The next step is to notice that $\lim_{n\to\infty} u_n(t) = 0$, for any $t \in T$. Indeed, fixing $t \in T$ and $\varepsilon > 0$, we get $p(t,\varepsilon) \in \mathbb{N}$ such that $\|f_p(t) - f(t)\| < \varepsilon$, for any $p \ge p(t,\varepsilon)$, because $f_n \underset{n}{\rightarrow} f$ (pointwise). Consequently, $u_p(t) \le \varepsilon$, for such p.

Finally, because ρ *a.c.*, it follows that $\rho(u_n) \underset{n}{\rightarrow} 0$, and this implies $\rho|f - f_n| \underset{n}{\rightarrow} 0$, because $|f - f_n| \le u_n$. □

Remark 3.3. It is not possible to drop the condition ρ *a.c.*, as the following counterexample shows.

Counterexample 3.4. Take $(T, \mathcal{T}, \mu) = (\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$, where μ is the counting measure. Take ρ given via $\rho(u) = \sup_n u(n)$. One can see that ρ is not *a.c.*, because $\rho(u_n) = 1$, for any $n \in \mathbb{N}$, where $u_n(m) = 0$, if $m \le n$ and $u_n(m) = 1$, if $m > n$ (clearly $u_n \downarrow 0$). Here classes and functions coincide ($\mathcal{L}_\rho(X) = L_\rho(X)$), because the only negligible set is ϕ.

Take $X = l^2$ with the canonical Schauder basis. Take $f \in \mathcal{L}_\rho(X)$, given via $f(m) = e_m$.

Then $f = \sum_{i=1}^{\infty} a_i e_i$, $a_i = \varphi_{\{i\}}$, $f_n = \sum_{i=1}^{n} a_i e_i$ and $\rho|f - f_n| = 1$, for any n.

Hence "$f_n \underset{n}{\rightarrow} f$ in $\mathcal{L}_\rho(X)$" is false.

We would like to close with the following:

Concrete example.

Again $(T, \mathcal{T}, \mu) = (\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$, where μ is the counting measure (all the functions are μ-measurable).

We shall consider the function norm $\rho = \| \ \|_1$, i.e.

$$\rho(u) = \sum_{n=1}^{\infty} u(n),$$

for any $u : \mathbb{N} \to \overline{\mathbb{R}}_+$.

Of course, ρ is *a.c.*

The Banach space X will be a Lorentz sequence space defined as follows:

Take a numerical sequence $w = (w_n)_n$ such that $w_n \downarrow 0$ and $\sum_{n=1}^{\infty} w_n = \infty$ (e.g. $w_n = \frac{1}{n}$).

Take $1 \leq p < \infty$. The function norm $\rho_1 : M_+(\mu) \to \overline{\mathbb{R}}_+$ (having the Riesz-Fischer property) given via

$$\rho_1(u) = \sup \left(\sum_{n=1}^{\infty} u(\pi(n))^p w_n\right)^{\frac{1}{p}}$$

(the supremum being taken over all bijections $\pi : \mathbb{N} \to \mathbb{N}$) generates

$$\mathcal{L}_{\rho_1} \equiv L_{\rho_1} \overset{not}{=} d(w, p).$$

Namely $d(w, p) = \{u : \mathbb{N} \to K \mid \rho_1 |u| < \infty\}$ which is a Banach space, called *Lorentz sequence space.*

This space admits the canonical Schauder basis as monotone Schauder basis and $\|e_n\| = w_1^{\frac{1}{p}}$, for any n.

Because $d(w, p) \subset c_0$= the sequences which have zero limit, one can compute the norm in $d(w, p)$ via

$$\|u\| = \rho_1(u) = \left(\sum_{n=1}^{\infty} (|u(n)|^*)^p w_n\right)^{\frac{1}{p}},$$

where $(|u(n)|^*)_n$ is the decreasing rearrangement of $(|u(n)|)_n$.

As anounced, we shall take $X = d(w, p)$, with $1 \leq p < \infty$ arbitrary and $w = (w_n)_n$, $w_n = \frac{1}{n}$, hence we are concerned with $\mathcal{L}_\rho(d(w, p))$.

We shall study a particular $f \in \mathcal{L}_\rho(d(w, p))$ in order to perform a numerical example.

Let $t \in K$ with $|t| < 1$.

We shall define the function $f : \mathbb{N} \to d(w, p)$ (see later), via

$$f(m) = a_m = (a_{m,n})_{n \geq 1}.$$

Here, for any $m \in \mathbb{N}$, we have: $a_{m,n} = 0$, if $1 \le n \le m$ and $a_{m,n} = t^n$, if $n > m$.

Via computation, one can see that (pointwise) $f = \sum_{n=1}^{\infty} F_n e_n$ (where $F_n : \mathbb{N} \to K$, $F_n(m) = a_{m,n}$) and $f_n = \sum_{i=1}^{n} F_i e_i$, for any n.

Indeed, one has $f(m) = a_m \in d(w,p)$, namely $a_m = \sum_{n=1}^{\infty} a_{m,n} e_n$, for any m. This is seen as follows. Because

$$(|a_{m,n}|^*)_{n\in\mathbb{N}} = (|t|^{m+1}, |t|^{m+2}, ...),$$

it follows that

$$\|a_m\|^p = \sum_{n=1}^{\infty} (|a_{m,n}|^*)^p w_n \le \frac{|t|^{mp}|t|^p}{1-|t|^p}.$$

On the other hand, the series $\sum_{n=1}^{\infty} a_{m,n} e_n = \sum_{n=m+1}^{\infty} a_{m,n} e_n$ converges absolutely, because $\|a_{m,n} e_n\| = |t^n| w_1^{\frac{1}{p}}$, if $n > m$ and its sum is precisely $a_m = f(m)$, due to the fact that (for $n > m$)

$$\left\| a_m - \sum_{i=1}^{n} a_{m,i} e_i \right\| = \sum_{i=n+1}^{\infty} |t|^{ip} \frac{1}{i-n} \le \sum_{i=n+1}^{\infty} |t|^{ip} \underset{n}{\to} 0.$$

It is also seen that $f \in \mathcal{L}_\rho(d(w,p))$, because

$$\rho|f| = \sum_{m=1}^{\infty} \|a_m\| \le \frac{|t|}{(1-|t|^p)^{\frac{1}{p}}} \sum_{m=1}^{\infty} |t|^m = \frac{|t|^2}{(1-|t|^p)^{\frac{1}{p}}(1-|t|)}.$$

After some (long) computations we obtain the following evaluation, valid for any $n \in \mathbb{N}$:

$$\rho|f - f_n| \le n|t|^{n+1} \frac{1}{(1-|t|^p)^{\frac{1}{p}}(1-|t|)}. \tag{2}$$

Let us prove (2). According to the previous computation, one has (pointwise) for any m and n,

$f - f_n = \sum_{i=n+1}^{\infty} F_i e_i \Rightarrow (f - f_n)(m) = \sum_{i=n+1}^{\infty} a_{m,i} e_i \Rightarrow |f - f_n|(m) =$
$= \|(0, 0, ..., 0, a_{m,n+1}, a_{m,n+2}, ...)\|$,
where, for any $h \ge 1$, one has $a_{m,n+h} = 0$ (if $n+h \le m$, i.e. $m > n$) and $a_{m,n+h} = t^{n+h}$ (if $n + h > m$, i.e. $1 \le m \le n$).

In order to compute $\rho|f - f_n| = \sum_{m=1}^{\infty} |f - f_n|(m)$, we must compute $(|f - f_n|(m))^p$ for any m and n. Let us fix n.

For $1 \le m \le n$, i.e. $n+h > m$, one has $a_{m,n+h} = t^{n+h}$, for any $h \ge 1$ and the decreasing rearrangement of $(0, 0, ..., 0, |a_{m,n+1}|, |a_{m,n+2}|, ...)$ is precisely $(|t|^{n+1}, |t|^{n+2}, ...)$, hence $(|f - f_n|(m))^p = \sum_{h=1}^{\infty} (|t|^{n+h})^p \frac{1}{h} \le |t|^{np} \frac{|t|^p}{1 - |t|^p}$ and this implies

$$\sum_{m=1}^{n} |f - f_n|(m) \le n|t|^n \frac{|t|}{(1 - |t|^p)^{\frac{1}{p}}}. \tag{3}$$

For $m > n$, i.e. $n+h \le m$, one has $a_{m,n+h} = 0$, hence, writing $i = n+h$, we have $(f - f_n)(m) = \sum_{i=n+1}^{\infty} a_{m,i} e_i = (0, 0, ..., 0, a_{m,n+1}, a_{m,n+2}, ...) =$
$= (0, 0, ..., 0, ..., 0, a_{m,m+1}, a_{m,m+2}, ...)$.

The decreasing rearrangement of the last sequence is $(|t|^{m+1}, |t|^{m+2}, ...)$, hence

$$(|f - f_n|(m))^p = \sum_{h=1}^{\infty} (|t|^{m+h})^p \frac{1}{h} \le |t|^{mp} \frac{|t|^p}{1 - |t|^p}.$$

We get

$$\sum_{m=n+1}^{\infty} |f - f_n|(m) \le \frac{|t|}{(1 - |t|^p)^{\frac{1}{p}}} \sum_{m=n+1}^{\infty} |t|^m = \frac{|t|}{(1 - |t|^p)^{\frac{1}{p}}} \cdot \frac{|t|^{n+1}}{1 - |t|}. \tag{4}$$

From (3) and (4) we obtain

$$\rho|f - f_n| = \sum_{m=1}^{\infty} |f - f_n|(m) \le \frac{|t|^{n+1}}{(1 - |t|^p)^{\frac{1}{p}}} (n + \frac{|t|}{1 - |t|}) \le$$
$$n|t|^{n+1} \frac{1}{(1 - |t|^p)^{\frac{1}{p}}} (1 + \frac{|t|}{1 - |t|}) = n|t|^{n+1} \frac{1}{(1 - |t|^p)^{\frac{1}{p}} (1 - |t|)}.$$

Of course, $\rho|f - f_n| \underset{n}{\to} 0$.

Take $\varepsilon > 0$ arbitrarily. We want to evaluate n in order to have

$$\rho|f - f_n| < \varepsilon. \tag{5}$$

It will be sufficient to have

$$n|t|^{n+1} \frac{1}{(1 - |t|^p)^{\frac{1}{p}} (1 - |t|)} < \varepsilon. \tag{6}$$

With an extra effort, one can prove that, in order to have (6), it will be sufficient to have

$$n > \frac{2|t|^2}{(1 - |t|^p)^{\frac{1}{p}} (1 - |t|)^3} \cdot \frac{1}{\varepsilon} - 1 \tag{7}$$

(inverse dependence upon ε).

Concrete numerical example.

Take $t = \frac{1}{2}$, $p = 2$ and $\varepsilon = 0,01 = \frac{1}{100}$.

We have

$$\frac{2|t|^2}{(1-|t|^p)^{\frac{1}{p}}(1-|t|)^3}\cdot\frac{1}{\varepsilon}-1=\frac{800}{\sqrt{3}}-1\approx 460,88021.$$

So, for $n \geq 461$, one has $\rho|f-f_n| < \frac{1}{100}$.

Conclusion. Using the general procedure exhibited above, one needs to perform at least 461 computational steps in order to obtain a two decimals precision.

Remark. The condition 7, which is very confortable to be applied, was obtained from 6, writing $|t| = \frac{1}{1+a}$, with $a > 0$ and majorizing

$$n|t|^{n+1}=\frac{n}{(1+a)^{n+1}}<\frac{n}{\frac{(n+1)n}{2}a^2}=\frac{2}{(n+1)a^2}=\frac{2}{n+1}\cdot\frac{|t|^2}{(1-|t|)^2}.$$

This procedure induces (unfortunately) an augmentation of the critical level for n, because the computation is not very sharp.

This can be seen using 6 and working directly. One must have

$$n|t|^{n+1}\cdot\frac{1}{(1-|t|^p)^{\frac{1}{p}}(1-|t|)}<\varepsilon=\frac{1}{100}.$$

Let us work for $t = \frac{1}{2}$, $p = 2$ and take $\varepsilon = \frac{1}{100}$. We obtain

$$\frac{n}{2^{n+1}}\cdot\frac{1}{\frac{\sqrt{3}}{2}\cdot\frac{1}{2}}=\frac{n}{2^{n-1}}\cdot\frac{1}{\sqrt{3}}<\frac{1}{100}.$$

It will be sufficient to have

$$\frac{n}{2^{n-1}}\cdot\frac{1}{\frac{3}{2}}<\frac{1}{100}\Leftrightarrow\frac{n}{2^{n-2}}\cdot\frac{1}{3}<\frac{1}{100}.$$

Because

$$2^{n-2}=(1+1)^{n-2}=1+(n-2)+\frac{(n-3)(n-2)}{2}+\ldots>$$
$$>1+(n-2)+\frac{(n-3)(n-2)}{2}=n-1+\frac{(n-3)(n-2)}{2},$$

we have

$$\frac{n}{2^{n-2}}\cdot\frac{1}{3}<\frac{n}{n-1+\frac{(n-3)(n-2)}{2}}\cdot\frac{1}{3}=\frac{2n}{3n^2-9n+12}$$

and it will be sufficient to have

$$\frac{2n}{3n^2-9n+12}<\frac{1}{100}\Leftrightarrow 3n^2-209n+12>0.$$

Because $\Delta = 209^2 - 12^2 = 43537$ it will be sufficient to have

$$n > \frac{209 + \sqrt{43537}}{6} \approx 69,609203.$$

For $n \geq 70$, one has

$$\rho|f - f_n| < \frac{1}{100}.$$

Conclusion. Working for this particular case and using the special procedure exhibited above, we need to perform only 70 computational steps in order to obtain again two decimals precision.

Final conclusion. Our paper is dedicated to the general idea of Numerical Analysis: approximating mathematical objects via finite objects of the same kind. More precisely, here we use the canonical finite dimensional subspaces generated by a Schauder basis to approximate all the vectors in the space. We obtained a general approximation formula.

It is easy to apply this general formula. Unfortunately, the results thus obtained are not very sharp (for our example, the obtained critical level was 461). Using a particular (specifical) method for this example, we obtained the critical level 70, which is by far better than the "general" level 461, furnishing a considerable more economic computation. This fact is characteristic for Numerical Analysis.

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