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Row and Column Representatives in Qualitative Analysis of Arbitrary Switching Positive Systems

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Abstract. Consider an arbitrary switching positive system whose (discrete- or continuous-time) dynamics is defined by the matrix set $\mathcal{A}$. The class of copositive Lyapunov functions called "polyhedral" is characterized by necessary and sufficient conditions derived from the solvability of some inequalities built with the matrices in $\mathcal{A}$, and from the properties of the row representatives of $\mathcal{A}$. Duality is proven between the "polyhedral" class and the already known class of "linear" copositive Lyapunov functions (characterized by equivalent conditions derived from the properties of the column representatives of $\mathcal{A}$). A numerical example illustrates the utilization of two types of approaches, as well as the connection among them.

Key-words: positive systems, switching systems, stability analysis.

1. Introduction

Notations and nomenclature used for the presentation of this research are provided in the *Appendix*.

1.1. Research context

Consider a family of N positive *linear time-invariant* (LTI) systems

$$\Sigma_{A^k} : x'(t) = A^k x(t),\ \ x(t_0) = x_0 \in \mathbb{R}^n_+;\ t_0, t \in \mathbb{T},\ k = 1, \ldots, N, \qquad (1)$$

with *continuous-time* (CT) dynamics, $\mathbb{T} = \mathbb{R}_+$, or *discrete-time* (DT) dynamics, $\mathbb{T} = \mathbb{Z}_+$, and the operator ( )' acting accordingly. Matrices $A^k \in \mathbb{R}^{n\times n}$ are essentially nonnegative and Hurwitz stable in CT case; nonnegative and Schur stable in DT case. Systems (1) describe the dynamics of the arbitrary switching positive system:

$$\Sigma_{\mathcal{A}}: \;\; x'(t) = A^{\upsilon(t)}x(t), \;\; x(t_0) = x_0 \in \mathbb{R}^n_+, \; t \geq t_0, \;\; \mathcal{A} = \left\{A^1, \dots, A^N\right\}, \quad (2)$$

where $\upsilon : \mathbb{T} \to \{1, \dots, N\}$ is the switching signal (piecewise continuous in CT case). LTI systems (1) are called the "constituent systems" of system (2).

For the class of systems defined above by (1) and (2), several papers, such as [9], [7], [8], [3], [10], [5], [6], have investigated Lyapunov function candidates defined by linear copositive functions

$$\overline{\mathcal{L}}: \; \mathbb{R}^n_+ \to \mathbb{R}_+, \; \overline{\mathcal{L}}(x) = \sum_{i=1}^{n} u_i x_i = u^T x, \;\; u = [u_1 \dots u_n]^T \gg 0. \quad (3)$$

Subsequently, the following noticeable result is available (see Theorem 4 in [7], Theorem 25.16 in [10] for CT case, and Theorem 1 in [6] for DT case), which points out the role of the column-structured strong linear inequalities built with the matrices of $\mathcal{A}$

$$u^T A^k \ll 0, k = 1, \dots, N, \quad \text{(4-CT)}$$

$$u^T (A^k - I) \ll 0, k = 1, \dots, N, \quad \text{(4-DT)}$$

as well as the role of the column representatives of matrix set $\mathcal{A}$ (see Appendix). All these column representatives $\bar{A}_\sigma \in \mathbb{R}^{n\times n}$, $\sigma \in \mathcal{C}$, are essentially nonnegative matrices in the CT case, and nonnegative matrices in the DT case, respectively. Subsequently, their Perron-Frobenius eigenvalues $\lambda_{\max}(\bar{A}_\sigma)$ are well defined, and there exists a dominant eigenvalue:

$$\bar{\lambda}^* = \max_{\sigma\in\mathcal{C}} \lambda_{\max}(\bar{A}_\sigma). \quad (5)$$

Theorem 1. [Characterization of linear copositive Lyapunov functions [7], [10], [6]]

*Statements **(i)** – **(iv)** given below are equivalent.*

(i) *There exist common Lyapunov functions of form* (3) *for the constituent systems* (1).

(ii) *The column-structured strong linear inequalities* (4-CT) *and* (4-DT), *respectively, have positive solutions* $u \gg 0$.

(iii) $\bar{\lambda}^* < 0$ (CT), *respectively* $\bar{\lambda}^* < 1$ (DT).

(iv) *All column representatives of the matrix set* $\mathcal{A}$ *defined by* (2) *are Hurwitz stable* (CT), *respectively Schur stable* (DT).

In our paper [12] we have shown that statements (i)-(iv) of Theorem 1 are also equivalent to the solvability of the column-structured weak quasi-linear inequalities

$$u^T A^k \leq r u^T, k = 1, \ldots, N, \tag{6}$$

where $u \gg 0$ is a positive vector and $r < 0$ (CT), respectively $0 < r < 1$ (DT) is a constant expressing the decreasing rate of the linear copositive Lyapunov function (3).

Within the context of the results summarized by Theorem 1, Example 1 in paper [8] studies the CT case of systems (2), defined by the matrices

$$A^1 = \begin{bmatrix} -1 & 2 \\ 1 & -3 \end{bmatrix}, \ A^2 = \begin{bmatrix} -6 & 6 \\ 2 & -6 \end{bmatrix}, \tag{7}$$

and proves that condition (4-CT) cannot be satisfied. The cited paper also shows that the positive vector $u = [5 \ \ 2]^T$ verifies the CT case of the following row-structured strong linear inequalities

$$A^k u \ll 0, k = 1, \ldots, N \tag{8-CT}$$

for which the DT counter-part means the inequalities

$$(A^k - I)u \ll 0, k = 1, \ldots, N \tag{8-DT}$$

Despite the similarity between condition (4-CT) and condition (8-CT), the cited paper [8] gives no system theoretic interpretation to inequalities (8-CT). Moreover, the cited paper claims that "for switched systems (in contrast with the LTI case), these two conditions, *i.e.* (4-CT) and (8-CT), are not equivalent". This appears as an intriguing statement, once the properties exhibited by LTI systems are expected to get generalized forms when operating with families of LTI systems.

1.2. Paper objectives and organization

Our paper builds a broader framework for the qualitative analysis of switching systems (2), which is able to accommodate Theorem 1 and, concomitantly, to reveal the role of the row-structured strong linear inequalities (8-CT) // (8-DT) - that remained hidden for paper [8].

Our research covers both CT and DT cases of switching system (2), focusing on two main objectives. The key instruments of the investigation are based on the properties of row and column representatives of the matrix set $\mathcal{A}$ defined by (2).

The first main objective is the study of the class of copositive Lyapunov functions called "polyhedral" in paper [2] and "max-type" in our earlier work [11], which are defined as

$$\underline{\mathcal{L}} : \mathbb{R}^n_+ \to \mathbb{R}_+, \underline{\mathcal{L}}(x) = \max_{i=1,\ldots,n} \left\{ \tfrac{x_i}{u_i} \right\}, \quad u = [u_1 \ldots u_n]^T \gg 0, \tag{9}$$

This study envisages two specific goals, namely:

- Characterization of Lyapunov functions (9) by a set of equivalent conditions expressed in terms of (u, r) pairs (with $u = [u_1 \dots u_n]^T \gg 0$ and $r < 0$ (CT), respectively $0 < r < 1$ (DT)), which solve the row-structured weak quasi-linear inequalities

$$A^k u \leq ru, k = 1, \dots, N, \tag{10}$$

- Existence of Lyapunov functions (9) correlated to the Perron Frobenius eigenvalues of the row representatives of the matrix set $\mathcal{A}$ defined by (2).

The second main objective is the duality analysis between "polyhedral" (9) and "linear" (3) copositive Lyapunov functions.

The remainder of the text is organized as follows. Sections 2 and 3 present the results supporting the two main objectives, respectively. Section 4 illustrates the applicability of these results to the switching system considered in [8], which is defined by the matrices reproduced in equation (7). For space reasons, complete proofs cannot be given for the theorems presented by the current article; therefore we just give brief indications for proof construction that rely on adequate adaptations (*mutatis mutandis*) of some previous results of ours, such as in [12], [11].

2. Characterization of polyhedral copositive Lyapunov functions

The analysis to be developed by the current section for polyhedral copositive Lyapunov functions is founded on a set of equivalent properties that are able to outline the role of the specific algebraic instruments in exploring the dynamics of switching system (2).

2.1. Use of row-structured inequalities

Theorem 2. [Characterization in terms of row-structured inequalities]

Let $u = [u_1 \dots u_n]^T \gg 0$, $||u||_\infty = 1$, be a vector and $r < 0$ (CT), respectively $0 < r < 1$ (DT), a constant.

The following five statements are equivalent.

(i) *The polyhedral copositive function* (9) *is a common Lyapunov function for the constituent systems* (1) *and it satisfies the inequality*

$$D_t^+ \underline{\mathcal{L}}(x(t)) = \lim_{h \downarrow 0} \tfrac{1}{h} \left(\underline{\mathcal{L}}(x(t+h)) - \underline{\mathcal{L}}(x(t)) \right) \leq r \underline{\mathcal{L}}(x(t)), // \tag{11-CT}$$

$$\underline{\mathcal{L}}(x(t+1)) \leq r \underline{\mathcal{L}}(x(t)), \tag{11-DT}$$

along each nontrivial trajectory of the arbitrary switching system (2).

(ii) *The contractive sets*

$$\underline{\mathcal{X}}(\varepsilon; t, t_0) = \left\{ x \in \mathbb{R}^n_+ | x_i \leq \varepsilon u_i e^{r(t-t_0)}, i = 1, \ldots, n \right\}, \varepsilon > 0, \; t, t_0 \in \mathbb{R}_+, // \tag{12-CT}$$

$$\underline{\mathcal{X}}(\varepsilon; t, t_0) = \left\{ x \in \mathbb{R}^n_+ | x_i \leq \varepsilon u_i \, r^{(t-t_0)}, i = 1, \ldots, n \right\}, \; \varepsilon > 0, \; t, t_0 \in \mathbb{Z}_+, \tag{12-DT}$$

are invariant with respect to the trajectories of the arbitrary switching system (2)*.*

(iii) *The following matrix inequalities hold true for* $U = diag\{u_1, \ldots, u_n\}$*:*

$$\mu_\infty(U^{-1} A^k U) \leq r, k = 1, \ldots, N, // \tag{13-CT}$$

$$||U^{-1} A^k U||_\infty \leq r, k = 1, \ldots, N. \tag{13-DT}$$

(iv) *The pair* (u, r) *satisfies the row-structured weak quasi-linear inequalities* (10)*.*

(v) *The row-structured strong linear inequalities* (8-CT) // (8-DT) *have positive solutions.*

Proof. We notice that function $\underline{\mathcal{L}}$ defined by (9) can be written as $\underline{\mathcal{L}}(x) = ||U^{-1}x||_\infty$. Similarly, the contractive sets defined by (12-CT) // (12-DT) can be described as

$$\underline{\mathcal{X}}(\varepsilon; t, t_0) = \left\{ x \in \mathbb{R}^n_+ | \;\; ||U^{-1}x||_\infty \;\; \leq \varepsilon e^{r(t-t_0)} \right\}, \varepsilon > 0, \; t, t_0 \in \mathbb{R}_+, // \tag{14-CT}$$

$$\underline{\mathcal{X}}(\varepsilon; t, t_0) = \left\{ x \in \mathbb{R}^n_+ | \;\; ||U^{-1}x||_\infty \;\; \leq \varepsilon \, r^{(t-t_0)} \right\}, \varepsilon > 0, \; t, t_0 \in \mathbb{Z}_+. \tag{14-DT}$$

We can use the results presented by Proposition 1 in our previous work [12] (that refers to function $\overline{\mathcal{L}}(x)$ of form (3) and column-structured inequalities); after appropriate transformations these results may handle weighted matrix norms induced by the vector norm $|| \;\; ||_\infty$. The connections between points (i), (ii) - expressing dynamic properties of the system trajectories, and points (iii), (iv), (v) - expressing algebraic properties of the system matrices, are addressed along the key lines, *mutatis mutandis*, which guide the proof of Theorem 3 in [12]. □

Remark 1. [Weak quasi-linear versus strong linear inequalities] *Points (iv) and (v) of Theorem 2 have been separately formulated, despite their similarity, because we aim at an individual targeting of strong linear inequalities of type* (8-CT) // (8-DT)*, whose implication in stability testing remained hidden for paper* [8]*, as mentioned in the introductory section.*

2.2. Use of row representatives

Based on the above result, we are able to reveal the role of the row representatives of the matrix set $\mathcal{A}$ defined by (2). For any label $\sigma \in \mathcal{C}$, the row representative $\underline{A}_\sigma \in \mathbb{R}^{n\times n}$ is (essentially) nonnegative and the Perron-Frobenius eigenvalue $\lambda_{\max}(\underline{A}_\sigma)$ is well defined (see Appendix); therefore we can introduce

$$\underline{\lambda}^* = \max_{\sigma\in\mathcal{C}} \lambda_{\max}(\underline{A}_{\,\sigma}), \underline{\mathcal{C}}^* = \{\sigma^* \in \mathcal{C} \;|\lambda_{\max}(\underline{A}_{\,\sigma^*}) = \underline{\lambda}^*\}\,, \tag{15}$$

where $\underline{\lambda}^*$ represents the dominant eigenvalue of the row representatives of $\mathcal{A}$ defined by (2), and $\underline{\mathcal{C}}^*$ is the *set of labels* associated with $\underline{\lambda}^*$.

Theorem 3. [Characterization in terms of row representatives]

The following statements are equivalent:

(i) *There exist* $u = [u_1 \dots u_n]^T \gg 0$, $||u||_\infty = 1$, *and* $r < 0$ (CT), *respectively* $r < 1$ (DT) *that satisfy statements **(i)**–**(v)** of Theorem 2.*

(ii) $\underline{\lambda}^* < 0$ (CT), *respectively* $\underline{\lambda}^* < 1$ (DT).

(iii) *The row representatives of* $\mathcal{A}$ *are Hurwitz stable* (CT), *respectively Schur stable* (DT).

Proof. We can use the results presented by Propositions 2, 3 and 4 in our previous work [12] (referring to the column representatives of $\mathcal{A}$ (2), which are associated with function $\overline{\mathcal{L}}(x)$ of form (3)); after appropriate transformations these results may handle the row representatives of $\mathcal{A}$ (2). Then, we exploit the dominant eigenvalue defined by (15) to show that the inequality $\underline{\lambda}^* < 0$ (CT case), and $\underline{\lambda}^* < 1$ (DT case), respectively, represents a necessary and sufficient condition for the solvability of inequalities (10). □

Remark 2. [Row- versus column-representatives] *Theorem 3 provides conditions for the existence of polyhedral copositive Lyapunov functions (relying on the row representatives of* $\mathcal{A}$*), which are of the same type as those known from literature for linear copositive Lyapunov functions (relying on the column representatives of* $\mathcal{A}$*). In other words, at the level of the whole section, the equivalent statements of our Theorems 2 and 3 create a scenario of similar nature, but complementing the requirements of Theorem 1 (that was presented in the introductory part of the article as the starting point for our research).*

Remark 3. [Construction based on Perron-Frobenius eigenstructure] *In the absence of a detailed proof for Theorem 3, we just mention that the construction of polyhedral copositive Lyapunov functions can use the Perron-Frobenius theory applied to the row representatives. Propositions 3 and 4 in* [12]*, transformed so as to operate on row- instead of column-representatives, prove that the Perron-Frobenius eigenvector associated with the dominant eigenvalue* (15) *may define a Lyapunov function of form* (9)*, once condition (ii) of Theorem 3 is fulfilled.*

3. Duality of the "polyhedral" and "linear" approaches

Consider the dual system of the arbitrary switching system (2), described by

$$\begin{array}{c} \Sigma_{\mathcal{B}}: \quad y'(t) = B^{v(t)}y(t),\ y(t_0) = y_0 \in \mathbb{R}^n_+,\ t \geq t_0, \\ \mathcal{B} = \left\{B^1, \ldots, B^N\right\}.\ B^k = (A^k)^T, k = 1, ..., N. \end{array} \tag{16}$$

Theorem 4. [Duality result]

The polyhedral copositive function (9) *is a Lyapunov function for system* (2)*, if and only if the linear copositive function* (3) *is a Lyapunov function for dual system* (16)*.*

Proof. From Theorem 2, the existence of Lyapunov function (9) for system (2) is equivalent to "$u \gg 0$ solves $A^k u \ll 0$, $k = 1, \ldots, N$ (CT), respectively $(A^k - I)u \ll 0$, $k = 1, \ldots, N$ (DT)". From Theorem 1, the existence of Lyapunov function (3) for system (16) is equivalent to "$u \gg 0$ solves $u^T(A^k)^T \ll 0$, $k = 1, \ldots, N$ (CT), respectively $u^T((A^k)^T - I) \ll 0$, $k = 1, \ldots, N$ (DT)". □

Remark 4. [Stability test based on conditions (4-CT) // (4-DT) or (8-CT) // (8-DT)] *The proof of Theorem 4 guarantees that the same vector $v \gg 0$ can be used for the construction of the Lyapunov function* (9) *(respectively function* (3)*) for system* (2) *(respectively system* (16)*). As a matter of fact, if one can prove that either Theorem 1 or Theorem 3 works for the primal system* (2)*, then the global exponential stability is ensured for the equilibrium {0} of both the primal system* (2) *and the dual system* (16)*. In terms of dominant eigenvalues, this means that a test using either $\underline{\lambda}^* < 0$, (CT case), and $\underline{\lambda}^* < 1$ (DT case) respectively – associated with the row representatives, or $\bar{\lambda}^* < 0$ (CT case), and $\bar{\lambda}^* < 1$ (DT case), respectively – associated with the column representatives, ensures the stability of primal and dual systems, concomitantly. In other words, a stability test for system* (2) *must incorporate both conditions* (4-CT) // (4-DT) *and* (8-CT) // (8-DT)*, fact which elucidates the uncertain aspects of the remark in paper* [8] *(mentioned in the introductory section), where the comparative analysis between inequalities* (4-CT) // (4-DT) *and inequalities* (8-CT) // (8-DT) *was incomplete.*

4. Illustrative example

Consider the switching system (2) defined by matrices A^1, A^2 reproduced by equation (7) in the introductory section, which were previously used by an example in paper [8]. The dominant eigenvalue of the row representatives is $\underline{\lambda}^* = -0.2679$, corresponding to matrix $\underline{A}_{\sigma^*} = A^1$ labeled by $\sigma^* = (1, 1)$. Subsequently, one can use Theorem 3 and Remark 4 in order to prove that the equilibrium {0} is globally asymptotically stable for both the primal system (2) and the dual system (15). Obviously, this result balances the inefficiency of the test based on $\bar{\lambda}^*=0$, which simply shows (as commented in paper [8]) that Theorem 1 fails as a sufficient condition applied to the considered system. Moreover, in accordance with Remarks 3 and 4, the Perron-Frobenius right eigenvector of matrix $\underline{A}_{\sigma^*} = A^1$, namely $u = [1\ \ 0.3660]^T$ corresponding to $\underline{\lambda}^*= -0.2679$, can be used to define the function $\underline{\mathcal{L}} : \mathbb{R}^2_+ \to \mathbb{R}_+$, $\underline{\mathcal{L}}(x) = \max\{x_1,\ x_2/0.3660\}$, which is a polyhedral copositive Lyapunov function for primal system (2), as well as the function $\overline{\mathcal{L}} : \mathbb{R}^2_+ \to \mathbb{R}_+$, $\overline{\mathcal{L}}(x) = x_1 + 0.3660x_2$, which is a linear copositive Lyapunov function for dual system (16). For the discussed vector u, one can simply verify that row-structured strong linear inequalities (8-CT) are satisfied.

5. Conclusion

Our results on Lyapunov functions for arbitrary switching positive systems yield a qualitative-analysis framework where the row- and column-representatives offer key instruments for dealing with two classes of candidates, namely (i) polyhedral and (ii) linear. The existence of Lyapunov functions belonging to class (i) (respectively (ii)) is related, by equivalence, to the placement of the dominant eigenvalue of the row (respectively column) representatives. If such Lyapunov functions exist, then their construction may use the right (respectively left) Perron-Frobenius eigenvector associated with the dominant eigenvalue. The duality of the approaches based on the two classes ensures the applicability of a result to the primal system, concomitantly with the applicability of the paired result to the dual system.

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Appendix (notations and nomenclature)

(Essentially) nonnegative and positive matrices [1]. • A rectangular matrix $X = [x_{ij}] \in \mathbb{R}^{n\times m}$ is called: *nonnegative* (notation $X \geq 0$) if $X \in \mathbb{R}^{n\times m}_+ \Leftrightarrow x_{ij} \geq 0$, $i = 1,\dots,n$, $j = 1,\dots,m$; *semi-positive* (notation$X > 0$) if $X \geq 0$ and $X \neq 0$; *positive* (notation$X \gg 0$) if $X \in Int(\mathbb{R}^{n\times m}_+) \Leftrightarrow x_{ij} > 0$, $i = 1,\dots,n$, $j = 1,\dots,m$. • A square matrix $X = [x_{ij}] \in \mathbb{R}^{n\times n}$ is called *essentially nonnegative* (*positive*) if $x_{ij} \geq 0$, $(x_{ij} > 0)$, $i,j = 1,\dots,n$, $i \neq j$.

Eigenstructure of (essentially) nonnegative matrices [1]. • Let $M \in \mathbb{R}^{n\times n}$ and let $\lambda_i(M)$, $i = 1,\dots,n$, denote its eigenvalues. ***M*** nonnegative has a real eigenvalue $\lambda_{\max}(M)$ such that $|\lambda_i(M)| \leq \lambda_{\max}(M)$, $i = 1,...,n$. ***M*** essentially nonnegative has a real eigenvalue $\lambda_{\max}(M)$, such that $Re\{\lambda_i(M)\} \leq \lambda_{\max}(M)$, $i = 1,...,n$. ***M*** (essentially) nonnegative has a nonnegative right eigenvector $w_R(M) > 0$, satisfying $||w_R(M)||_\infty = 1$, and a nonnegative left eigenvector $w_L(M) > 0$, satisfying $||w_L(M)||_\infty = 1$, that correspond to the eigenvalue $\lambda_{\max}(M)$. ***M*** (essentially) nonnegative and *irreducible* (*i.e.* its associated graph is strongly connected) has $\lambda_{\max}(M)$ simple eigenvalue and $w_R(M) \gg 0$, $w_L(M) \gg 0$ (Perron-Frobenius *eigenstructure*).

Matrix norms and measures [4]. Let $p \in \{1, \infty\}$ • For a vector $x = [x_1 \ldots x_n]^T \in \mathbb{R}^n$, $||x||_p$ denotes the Hölder vector p-norm; in particular $||\, x \,||_1 = \sum_{i=1}^n |\, x_i \,|$, $||\, x \,||_\infty = \max_{i=1,\ldots,n} \{|x_i|\}$. • For a matrix $M \in \mathbb{R}^{n \times n}$ $||M||_p$ denotes the matrix norm induced by the vector norm $||\ \ ||_p$; in particular $||\, M \,||_1 = \max_{1 \le j \le n} \{\sum_{i=1}^n |m_{ij}|\}$, $||\, M \,||_\infty = \max_{1 \le i \le n} \left\{\sum_{j=1}^n |m_{ij}|\right\}$. $\mu_p(M) = \lim_{h \downarrow 0} \frac{1}{h}(\,||I + hM||_p - 1)$ is the matrix measure based on the matrix norm $||\ \ ||_p$; in particular $\mu_1(M) = \max_{1 \le j \le n} \left\{m_{jj} + \sum_{i=1, i \ne j}^n |m_{ij}|\right\}$, $\mu_\infty(M) = \max_{1 \le i \le n} \left\{m_{ii} + \sum_{j=1, j \ne i}^n |m_{ij}|\right\}$.

Row / Column representatives of a set of matrices [13]. Let $\mathcal{M} = \left\{M^1, \ldots, M^N\right\} \subset \mathbb{R}^{n \times n}$ be a set of matrices. For a function $\sigma : \{1, \ldots, n\} \to \{1, \ldots, N\}$ denote by $\sigma = (\sigma(1), \ldots, \sigma(n))$ the corresponding n-tuple and denote by $\mathcal{C}$ the set of all the n-tuples with elements from $\{1, \ldots, N\}$.• For any $\sigma \in \mathcal{C}$, we construct the matrix $\underline{M}_\sigma \in \mathbb{R}^{n \times n}$, whose first row is the first row of $M^{\sigma(1)}$ (denoted as $[M^{\sigma(1)}]_{(1,:)}$), second row is the second row of $M^{\sigma(2)}$ (denoted as $[M^{\sigma(2)}]_{(2,:)}$), and so on, that is $\underline{M}_\sigma = \begin{bmatrix} [M^{\sigma(1)}]_{(1,:)} \\ \ldots \\ [M^{\sigma(n)}]_{(n,:)} \end{bmatrix} \in \mathbb{R}^{n \times n}$. A matrix $\underline{M}_\sigma \in \mathbb{R}^{n \times n}$, $\sigma \in \mathcal{C}$, constructed as above is called a *row representative* of the matrix set $\mathcal{M}$. We say that $\sigma \in \mathcal{C}$ labels the row representatives. • For any $\sigma \in \mathcal{C}$, we construct the matrix $\bar{M}_\sigma \in \mathbb{R}^{n \times n}$, whose first column is the first column of $M^{\sigma(1)}$ (denoted as $[M^{\sigma(1)}]_{(:,1)}$), second column is the second column of $M^{\sigma(2)}$ (denoted as $[M^{\sigma(2)}]_{(:,2)}$), and so on, that is $\bar{M}_\sigma = \left[[M^{\sigma(1)}]_{(:,1)} \ \ldots \ [M^{\sigma(n)}]_{(:,n)}\right] \in \mathbb{R}^{n \times n}$. A matrix $\bar{M}_\sigma \in \mathbb{R}^{n \times n}$, $\sigma \in \mathcal{C}$, constructed as above is called a *column representative* of the matrix set $\mathcal{M}$. We say that $\sigma \in \mathcal{C}$ labels the column representatives.

Acronyms. CT – continuous time, DT – discrete time, LTI – linear time invariant.

Notation. "X // Y" is used in place of "X [respectively Y]" for a compact writing of CT // DT statements.

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