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Interval Matrix Systems and Invariance of Non-Symmetrical Contractive Sets

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Abstract. The paper studies the set invariance properties exhibited by the linear dynamics of interval matrix systems (abbreviated IMSs). The considered sets have non-symmetrical shapes defined by arbitrary Hölder p-norms and are exponentially contractive (with exponential time-decrease). For such sets we provide two types of results that allow invariance tests: (i) sufficient conditions for $1 \le p \le \infty$, (ii) necessary and sufficient conditions for $p \in \{1, \infty\}$. Our approach includes, in generalized forms, results previously reported by other works referring to either non-symmetrical sets invariant with respect to single-model systems, or symmetrical sets invariant with respect to IMSs. Our presentation focuses on continuous-time IMSs, whereas the discrete-time case is briefly addressed by the key elements of a basic analogy. An example already used in the IMS literature illustrates the applicability of our theoretical developments.

Key-words: interval matrix systems, set invariance, non-symmetrical sets, contractive sets.

1. Introduction

1.1. Notations

Let $x = [x_1 \dots x_m]^\top \in \mathbb{R}^m$ be a vector. The Hölder vector p-norm is defined by $||x||_{p,m} = [|x_1|^p + \ldots + |x_m|^p]^{1/p}$ for $1 \le p < \infty$, and by $||x||_{\infty,m} = \max_{i=\overline{1,m}} |x_i|$ for $p = \infty$.

Let $A = [a_{ij}] \in \mathbb{R}^{m \times m}$ be a square matrix. The matrix norm induced by the vector p-norm is defined by $||A||_{p,m} = \max_{x \in \mathbb{R}^m,\ ||x||_{p,m}=1} ||Ax||_{p,m}$ for $1 \le p \le \infty$. The *matrix measure* of A is given by $\mu_{||\ ||_{p,m}}(A) = \lim_{h \searrow 0} \frac{1}{h}\left[||\, I_m + hA||_{p,m} - 1\right]$, where I_m stands for the unit matrix of order m. Fact 11.15.7 from [3] gives the expressions of the induced matrix norms and the corresponding matrix measures in the particular cases when $p \in \{1, 2, \infty\}$ as:

$$||A||_{1,m} = \max_{j=\overline{1,m}} \sum\nolimits_{i=1}^{m} |\, a_{ij}\,|,\ \ \mu_{||\ ||_{1,m}}(A) = \max_{j=\overline{1,m}} \left\{ a_{jj} + \sum_{i=1, i \ne j}^{m} |\, a_{ij}\,| \right\},$$

$$||\,A\,||_{2,m} = \left(\lambda_{\max}(A^T A)\right)^{1/2},\ \ \mu_{2,m}(A) = \tfrac{1}{2}\lambda_{\max}(A + A^T),$$

$$||\,A\,||_{\infty,m} = \max_{i=\overline{1,m}} \sum_{j=1}^{m} |\, a_{ij}\,|,\ \ \mu_{||\ ||_{\infty,m}}(A) = \max_{i=\overline{1,m}} \left\{ a_{ii} + \sum_{j=1,\ j \ne i}^{m} |\, a_{ij}\,| \right\}.$$

1.2. Paper goal and structure

The paper explores the invariance properties exhibited by the trajectories of *interval matrix systems* (abbreviated as IMSs) relative to several classes of non-symmetrical sets presenting an exponentially decreasing time-dependence (or, equivalently, an exponential contraction).

The IMS dynamics are described in continuous-time by

$$\dot{x}(t) = Ax(t), x(t_0) = x_0, t, t_0 \in \mathbb{R}_+, t \ge t_0, A \in \mathcal{A}, \tag{1}$$

where the set of square matrices

$$\begin{aligned} &\mathcal{A} = \left\{A \in \mathbb{R}^{n \times n} \middle|\ \underline{A} \le A \le \bar{A}\right\} = \left[\underline{A}, \bar{A}\right], \\ &\underline{A} = \left[\underline{a}_{ij}\right],\ A = [a_{ij}],\ \bar{A} = [\bar{a}_{ij}] \in \mathbb{R}^{n \times n}, \end{aligned} \tag{2}$$

is called an *interval matrix*.

An IMS of type (1) represents a multi-model system, which facilitates the mathematical description of linear dynamics affected by uncertainties, in the sense that the incomplete knowledge of values a_{ij} is addressed in terms of known lower and upper bounds $\underline{a}_{ij} \le a_{ij} \le \bar{a}_{ij}$. The trajectory of an IMS corresponding to a certain initial state $x_0 \in \mathbb{R}^n$ must be understood as the group of trajectories generated by $x_0 \in \mathbb{R}^n$ as initial condition for all single-model systems with $A \in \mathcal{A}$. Subsequently, the invariance properties of the IMS trajectories refer to the whole group of trajectories associated with any individual system whose matrix A belongs to the set $\mathcal{A}$.

Our study of IMS invariance focusses on contractive sets with non-symmetrical shapes, defined as follows:

Let $\|\cdot\|_{p,2n}$ be the Hölder p-norm in $\mathbb{R}^{2n}$, with $1 \leq p \leq \infty$, and for any vector $x = [x_1, ..., x_n]^T \in \mathbb{R}^n$ define a non-negative vector $\tilde{x} \in \mathbb{R}_+^{2n}$, as follows

$$\tilde{x} = \begin{bmatrix} x^+ \\ x^- \end{bmatrix} \in \mathbb{R}_+^{2n}, \text{ where } x^+ = \left[x_1^+, ..., x_n^+\right]^T, \; x^- = \left[x_1^-, ..., x_n^-\right]^T \in \mathbb{R}^n,$$
$$\text{with } \; x_i^+ = \max\{0, x_i\}, \;\; x_i^- = \max\{0, -x_i\}, \; i = \overline{1,n}. \tag{3}$$

Let $D^+ = diag\{d_1^+, ..., d_n^+\}$, $D^- = diag\{d_1^-, ..., d_n^-\} \in \mathbb{R}_+^{n\times n}$, with $d_i^+, d_i^- > 0$, $i = \overline{1,n}$, be two positive definite diagonal matrices, and

$$\widetilde{D} = \begin{bmatrix} D^+ & O_{n\times n} \\ O_{n\times n} & D^- \end{bmatrix} \in \mathbb{R}_+^{2n\times 2n}. \tag{4}$$

The sets described by

$$\mathcal{C}_p = \left\{ x \in \mathbb{R}^n | \left\| \widetilde{D}\tilde{x} \right\|_{p,2n} \leq 1 \right\} \subset \mathbb{R}^n, \tag{5}$$

are non-symmetrical and may have arbitrary shapes. They allow us to define the time-dependent sets

$$\mathcal{K}_p : \mathbb{R}_+ \rightrightarrows \mathbb{R}^n, \;\; \mathcal{K}_p(t) = e^{\alpha t}\mathcal{C}_p, \alpha \leq 0, \tag{6}$$

which are contractive for $\alpha < 0$, and constant for $\alpha = 0$. In accordance with [4], we say that the non-symmetrical set $\mathcal{K}_p(t)$ (6) is invariant with respect to (abbreviated w.r.t.) IMS (1), if any trajectory $x(t; t_0, x_0)$ of IMS (1) initialized at t_0 in $\mathcal{K}_p(t_0)$ does not leave $\mathcal{K}_p(t)$ for any $t \geq t_0$, i.e.

$$\forall t_0 \in \mathbb{R}_+, \forall x_0 \in \mathbb{R}^n, \left\|\widetilde{D}\widetilde{x}(t_0, x_0)\right\|_{p,2n} \leq 1 \; \Rightarrow$$
$$\Rightarrow \;\; \left\|\widetilde{D}\widetilde{x}(t, x(t; t_0, x_0))\right\|_{p,2n} \leq e^{\alpha(t-t_0)}, \; \forall t > t_0.$$

We also consider the non-symmetrical Lyapunov function candidates of form

$$\mathcal{V}_p : \mathbb{R}^n \to \mathbb{R}_+, \; \mathcal{V}_p(x) = \left\|\widetilde{D}\widetilde{x}\right\|_{p,2n}, \tag{7}$$

because the theory of contractive-set-invariance for IMSs is strongly related to the existence of common Lyapunov functions, as resulting from our works [13] and [14] (where the discussed invariant sets are symmetrical). Details on the general aspects of the connection between invariant sets and Lyapunov functions can be found in the monograph [4]. Besides the two papers mentioned above [13] and [14] (that focused on IMSs), the problem of common Lyapunov functions for classes of systems (not necessarily IMSs) was also discussed by [7], [10], [6], [1], [9], [11]. It is important to notice that the invariant sets, as well as the Lyapunov functions considered by all these cited papers have symmetrical forms.

In the current work devoted to IMS dynamics, the exploration of invariant sets with non-symmetrical forms (6) is inspired by the existence of some results referring to single-model systems, as in [2] and the generalization constructed in our work [12]. Relying on the background for single-model dynamics, we are able to approach the interval-model dynamics and provide invariance conditions for non-symmetrical sets $\mathcal{K}(t)$ of type (6); the conditions to be presented in the remainder of the text are sufficient for all $1 \leq p \leq \infty$ (part 1 of Section 2), and necessary and sufficient for $p \in \{1, \infty\}$ (part 2 of Section 2). Section 3 discusses several aspects related to the invariance of the non-symmetrical sets (6) w.r.t. IMS (1), such as: practical utilization of the conditions derived in Section 2; correlation between the non-symmetrical Lyapunov function candidates (7) and set-invariance; particular case of symmetrical invariant sets; approach to discrete-time IMSs. Section 4 illustrates the applicability of our theoretical developments, by the help of an example already used in the IMS literature.

2. Main results

2.1. Sufficient conditions for set invariance

Theorem 1. *Let* $1 \leq p \leq \infty$. *Consider an IMS of form* (1) *and a non-symmetrical time-dependent set* $\mathcal{K}_p(t)$ *of form* (6).

Introduce the set of matrices $\mathcal{U} = \left\{ \tilde{U} \in \mathbb{R}^{2n\times 2n} |\ \tilde{U} \text{ defined by Table 1} \right\}$, *where:*

$$\widetilde{U} = \begin{bmatrix} U^{11} & U^{12} \\ U^{21} & U^{22} \end{bmatrix} \in \mathbb{R}^{2n\times 2n},\ U^{11}, \ldots, U^{22} \in \mathbb{R}^{n\times n}, \tag{8}$$

and the entries of matrices $U^{11} = \left[u_{ij}^{11}\right]$, $U^{12} = \left[u_{ij}^{12}\right]$, $U^{21} = \left[u_{ij}^{21}\right]$, $U^{22} = \left[u_{ij}^{22}\right]$,$i, j = 1, \ldots, n$, *are given by Table 1.*

The non-symmetrical set (6) *is invariant w.r.t IMS* (1) *if*

$$\max_{\widetilde{U}\in\mathcal{U}} \mu_{\|\cdot\|_{p,2n}}(\widetilde{U}) \leq \alpha. \tag{9}$$

Proof. Let A be an arbitrary matrix satisfying $\underline{A} \leq A \leq \bar{A}$, and construct the essentially non-negative matrix

$$\widetilde{A} = \begin{bmatrix} \tilde{A}^{11} & \tilde{A}^{12} \\ \tilde{A}^{21} & \tilde{A}^{22} \end{bmatrix} \in \mathbb{R}^{2n\times 2n}, \tag{10}$$

where $\tilde{A}^{11} = \tilde{A}^{22} = A^{+} = \left[a_{ij}^{+}\right] \in \mathbb{R}^{n\times n}$, with $a_{ij}^{+} = \max\{0, a_{ij}\}$, $i \neq j$, $i, j = \overline{1,n}$, $a_{ii}^{+} = a_{ii}$, $i = \overline{1,n}$, and $\tilde{A}^{12} = \tilde{A}^{21} = A^{-} = \left[a_{ij}^{-}\right] \in \mathbb{R}^{n\times n}$, with $a_{ij}^{-} = \max\{0, -a_{ij}\}$, $i \neq j$, $i, j = \overline{1,n}$, $a_{ii}^{-} = 0$, $i = \overline{1,n}$ (in accordance with the utilization of the non-negative vector $\tilde{x} \in \mathbb{R}^{2n}$ (3) detailed by our previous work [12].

Table 1. (used by the statement of Theorem 1). Entries of matrices $U^{11} = \left[u_{ij}^{11}\right]$, $U^{12} = \left[u_{ij}^{12}\right]$, $U^{21} = \left[u_{ij}^{21}\right]$, $U^{22} = \left[u_{ij}^{22}\right]$, $i, j = 1, \dots, n$

i, j	Condition on interval $[\underline{a}_{ij}, \bar{a}_{ij}]$	u_{ij}^{11}	u_{ij}^{12}	u_{ij}^{21}	u_{ij}^{22}
$i = j$	$a_{ii} \in \mathbb{R}$	$\bar{a}_{ii}$	0	0	$\bar{a}_{ii}$
$i \neq j$	$0 \leq \underline{a}_{ij}$	$\bar{a}_{ij} \frac{d_i^+}{d_j^+}$	0	0	$\bar{a}_{ij} \frac{d_i^-}{d_j^-}$
	$\bar{a}_{ij} \leq 0$	0	$-\underline{a}_{ij} \frac{d_i^+}{d_j^-}$	$-\underline{a}_{ij} \frac{d_i^-}{d_j^+}$	0
	$\underline{a}_{ij} < 0 < \bar{a}_{ij}$	$\bar{a}_{ij} \frac{d_i^+}{d_j^+}$	0	0	$\bar{a}_{ij} \frac{d_i^-}{d_j^-}$
		OR			
		0	$-\underline{a}_{ij} \frac{d_i^+}{d_j^-}$	$-\underline{a}_{ij} \frac{d_i^-}{d_j^+}$	0

Hence, the essentially non-negative matrix $\tilde{D}\tilde{A}\tilde{D}^{-1} \in \mathbb{R}^{2n \times 2n}$ can be written in block form

$$\tilde{D}\tilde{A}\tilde{D}^{-1} = \begin{bmatrix} \left(\tilde{D}\tilde{A}\tilde{D}^{-1}\right)^{11} & \left(\tilde{D}\tilde{A}\tilde{D}^{-1}\right)^{12} \\ \left(\tilde{D}\tilde{A}\tilde{D}^{-1}\right)^{21} & \left(\tilde{D}\tilde{A}\tilde{D}^{-1}\right)^{22} \end{bmatrix} = \\ = \begin{bmatrix} (D^+)\tilde{A}^{11}(D^+)^{-1} & (D^+)\tilde{A}^{12}(D^-)^{-1} \\ (D^-)\tilde{A}^{21}(D^+)^{-1} & (D^-)\tilde{A}^{22}(D^-)^{-1} \end{bmatrix} = \\ = \begin{bmatrix} (D^+)A^+(D^+)^{-1} & (D^+)A^-(D^-)^{-1} \\ (D^-)A^-(D^+)^{-1} & (D^-)A^+(D^-)^{-1} \end{bmatrix}. \tag{11}$$

Table 2 presents the entries of the matrix $\tilde{D}\tilde{A}\tilde{D}^{-1}$ together with their majorants expressed in terms of the interval bounds $\underline{a}_{ij}, \bar{a}_{ij}$.

Table 2. (used by the proof of Theorem 1). Entries of the matrix $\tilde{D}\tilde{A}\tilde{D}^{-1}$ majorized in terms of the interval bounds $\underline{a}_{ij}, \bar{a}_{ij}$

i, j	Distinct cases for a_{ij}	$(\tilde{D}\tilde{A}\tilde{D}^{-1})_{ij}^{11} = \tilde{a}_{ij}^{11} \frac{d_i^+}{d_j^+}$	$(\tilde{D}\tilde{A}\tilde{D}^{-1})_{ij}^{12} = \tilde{a}_{ij}^{12} \frac{d_i^+}{d_j^-}$	$(\tilde{D}\tilde{A}\tilde{D}^{-1})_{ij}^{21} = \tilde{a}_{ij}^{21} \frac{d_i^-}{d_j^+}$	$(\tilde{D}\tilde{A}\tilde{D}^{-1})_{ij}^{22} = \tilde{a}_{ij}^{22} \frac{d_i^-}{d_j^-}$
$i = j$	$a_{ii} \in \mathbb{R}$	$a_{ii} \leq \bar{a}_{ii}$	0	0	$a_{ii} \leq \bar{a}_{ii}$
$i \neq j$	$0 \leq a_{ij}$	$a_{ij} \frac{d_i^+}{d_j^+} \leq \bar{a}_{ij} \frac{d_i^+}{d_j^+}$	0	0	$a_{ij} \frac{d_i^-}{d_j^-} \leq \bar{a}_{ij} \frac{d_i^-}{d_j^-}$
	$a_{ij} \leq 0$	0	$-a_{ij} \frac{d_i^+}{d_j^-} \leq -\underline{a}_{ij} \frac{d_i^+}{d_j^-}$	$-a_{ij} \frac{d_i^-}{d_j^+} \leq -\underline{a}_{ij} \frac{d_i^-}{d_j^+}$	0

Regardless of the concrete values of the entries a_{ij} of a matrix $A \in \mathcal{A}$, there exists a matrix $\tilde{U} \in \mathcal{U}$ of form (8) which majorizes the associated matrix $\tilde{D}\tilde{A}\tilde{D}^{-1}$. Since both $\tilde{D}\tilde{A}\tilde{D}^{-1}$ and $\tilde{U}$ are essentially non-negative, the componentwise inequality $\tilde{D}\tilde{A}\tilde{D}^{-1} \le \tilde{U}$ implies $\mu_{\|\cdot\|_{p,2n}}(\tilde{D}\tilde{A}\tilde{D}^{-1}) \le \mu_{\|\cdot\|_{p,2n}}(\tilde{U})$. Thus, for any $A \in \mathcal{A}$ the inequality $\mu_{\|\cdot\|_{p,2n}}(\tilde{D}\tilde{A}\tilde{D}^{-1}) \le \alpha$ is fulfilled in accordance with hypothesis (9), and Corollary 1 in [12] ensures the invariance of the non-symmetrical set $\mathcal{K}_p(t)$ of form (6) w.r.t. the trajectories of the single-model system defined by the considered matrix A. As $A \in \mathcal{A}$ is an arbitrary matrix, the invariance is guaranteed for all trajectories of IMS (1), once hypothesis (9) is satisfied. □

2.2. Necessary and sufficient conditions for set invariance

Theorem 2. *Consider an IMS of form* (1).

a) Let $p = 1$. Introduce the matrix

$$U_1 = \begin{bmatrix} U^{11} & 0 \\ 0 & U^{22} \end{bmatrix} \in \mathbb{R}^{2n\times 2n},$$
$$\left[U^{11}\right]_{ij} = \max\{\bar{a}_{ij}\tfrac{d_i^+}{d_j^+},\ -\underline{a}_{ij}\tfrac{d_i^-}{d_j^+}\}, i \ne j,\ \left[U^{11}\right]_{ii} = \bar{a}_{ii},$$
$$\left[U^{22}\right]_{ij} = \max\{-\underline{a}_{ij}\tfrac{d_i^+}{d_j^-},\ \bar{a}_{ij}\tfrac{d_i^-}{d_j^-}\}, i \ne j,\ \left[U^{22}\right]_{ii} = \bar{a}_{ii}. \tag{12}$$

The non-symmetrical set $\mathcal{K}_1(t)$ of form (6) *is invariant w.r.t IMS* (1) *if and only if*

$$\mu_{\|\cdot\|_{1,2n}}(U_1) \le \alpha. \tag{13}$$

b) Let $p = \infty$. Introduce the matrix

$$U_\infty = \begin{bmatrix} U^{11} & 0 \\ 0 & U^{22} \end{bmatrix} \in \mathbb{R}^{2n\times 2n},$$
$$\left[U^{11}\right]_{ij} = \max\{\bar{a}_{ij}\tfrac{d_i^+}{d_j^+},\ -\underline{a}_{ij}\tfrac{d_i^+}{d_j^-}\}, i \ne j,\ \left[U^{11}\right]_{ii} = \bar{a}_{ii},$$
$$\left[U^{22}\right]_{ij} = \max\{-\underline{a}_{ij}\tfrac{d_i^-}{d_j^+},\ \bar{a}_{ij}\tfrac{d_i^-}{d_j^-}\}, i \ne j,\ \left[U^{22}\right]_{ii} = \bar{a}_{ii}. \tag{14}$$

The non-symmetrical set $\mathcal{K}_\infty(t)$ of form (6) *is invariant w.r.t IMS* (1) *if and only if*

$$\mu_{\|\cdot\|_{\infty,2n}}(U_\infty) \le \alpha. \tag{15}$$

Proof. a) For $p = 1$ the following equality holds true

$$\max_{\widetilde{U}\in\mathcal{U}} \mu_{\|\cdot\|_{1,2n}}(\widetilde{U}) = \mu_{\|\cdot\|_{1,2n}}(U_1). \tag{16}$$

Sufficiency: From (13) and (16), hypothesis (9) of Theorem 1 is satisfied.

Necessity: Given an arbitrary matrix $A \in \mathcal{A}$, the set $\mathcal{K}_1(t)$ of form (6) is invariant w.r.t. the single-model system defined by A; Corollary 1 and Remark 1 in [12] yield the inequality $\mu_{\|\cdot\|_{p,2n}}(\tilde{D}\tilde{A}\tilde{D}^{-1}) \leq \alpha$. On the other hand, any matrix $\tilde{U} \in \mathcal{U}$ considered by Theorem 1 can be built as $\tilde{U} = \tilde{D}\tilde{A}\tilde{D}^{-1}$, corresponding to a certain $A \in \mathcal{A}$. In other words inequality (9) is satisfied, and, relying on equality (16), inequality (13) is satisfied too.

b) The proof is similar to the proof of a). □

3. Discussions

General exploitation of Theorems 1 and 2. If an interval matrix $A \in \mathcal{A}$ includes $q \leq n(n-1)$ bipolar intervals $[\underline{a}_{ij},\ \bar{a}_{ij}]$, $i \neq j$, with $\underline{a}_{ij} < 0 < \bar{a}_{ij}$, then the matrix set $\mathcal{U}$ requested by the hypothesys of Theorem 1 (for $1 < p < \infty$) contains 2^q distinct matrices $\tilde{U}$ of form (8). Apparently this fact could represent a disadvantage for applications, but one must take into account that the bipolar intervals $[\underline{a}_{ij},\ \bar{a}_{ij}]$ have rare motivation in practice; the uncertainty of a matrix entry regularly means unknown values, but with the same sign. Obviously, for an interval matrix without bipolar intervals, the matrix set $\mathcal{U}$ includes a single test matrix.

The particular forms of matrix measures corresponding to $p \in \{1, \infty\}$ simplifies hypothesis (9) of Theorem 1, so as Theorem 2 uses a single test matrix, regardless of the unipolar or bipolar form of the intervals defining $\mathcal{A}$ of the IMS. Theorem 2 (b) provides an equivalent form of Theorem 3 in our previous work [15] that was formulated as a necessary and sufficient condition for the componentwise exponential stability of IMSs. This equivalence is not an unexpected result, because the componentwise exponential stability represents a particular type of stability that individually constrains each state variable, by acting as an invariant contractive rectangular box, with exponential decrease.

Connections to non-symmetrical Lyapunov function candidates. The invariance of the non-symmetrical sets $\mathcal{K}_p(t)$ of form (6) w.r.t. IMS (1) is equivalent to the decrease of non-symmetrical functions $\mathcal{V}_p(x)$ of form (7) with the rate $\alpha \leq 0$ along any non-trivial trajectory of IMS (1), i.e.

$$\mathcal{D}_t^+ \mathcal{V}_p(x(t)) = \lim_{\theta \searrow 0} \tfrac{1}{\theta}[\mathcal{V}_p(x(t+\theta)) - \mathcal{V}_p(x(t))] \leq \alpha \mathcal{V}_p(x(t)). \tag{17}$$

This equivalence can be proven by considering the results available in [4] for single-model systems, applied to a generic system defined by a matrix $A \in \mathcal{A}$.

Thus, the current remark is able to generalize the existence of non-symmetrical Lyapunov functions initially studied in [2] for the discrete-time dynamics of single-model systems with $p \in \{1, 2, \infty\}$, to the continuous-time dynamics of IMSs with $1 \leq p < \infty$.

Particular case of symmetrical invariant sets. The invariant set candidates $\mathcal{K}_p(t)$ have form (6), with $\mathcal{C}_p$ (5) replaced by

$$\mathcal{C}_p = \left\{x \in \mathbb{R}^n | \, \|Dx\|_{p,n} \leq 1\right\} \subset \mathbb{R}^n, D = diag\{d_1, \ldots, d_n\} \in \mathbb{R}^n, \qquad (5')$$

where $d_i^- = d_i^+ = d_i > 0$. The test of Theorem 1 can be simplified accordingly, and condition (9) is replaced by

$$\mu_{\| \, \|_{p,n}}(DUD^{-1}) \leq \alpha, 1 \leq p \leq \infty, \qquad (9')$$

where

$$U = [u_{ij}] \in \mathbb{R}^{n \times n}, \;\; u_{ij} = \begin{cases} \max\{|\bar{a}_{ij}|, \; |\underline{a}_{ij}|\}, \text{ if } i \neq j, \\ \bar{a}_{ii} \;, \;\text{ if } i = j. \end{cases} \qquad (8')$$

Indeed, consider an arbitrary matrix $A \in \mathcal{A}$ and introduce the essentially non-negative matrix $\widehat{A} = [\hat{a}_{ij}] \in \mathbb{R}^{n \times n}$, $\hat{a}_{ij} = \begin{cases} |a_{ij}|, \text{ if } i \neq j \\ a_{ii} \;, \text{ if } i = j \end{cases}$. The matrix measure properties ensure the inequalities $\mu_{\| \, \|_{p,n}}(DAD^{-1}) \leq \mu_{\| \, \|_{p,n}}(D\hat{A}D^{-1})$ and $\mu_{\| \, \|_{p,n}}(D\hat{A}D^{-1}) \leq \mu_{\| \, \|_{p,n}}(U)$, showing that $\forall A \in \mathcal{A} : \mu_{\| \, \|_{p,n}}(DAD^{-1}) \leq \mu_{\| \, \|_{p,n}}(U)$.

For $p \in \{1, \infty\}$ inequality (9') also represents a necessary condition for the invariance of contractive symmetrical sets defined by (6) with (5'). Indeed, given an arbitrary matrix $A \in \mathcal{A}$, the invariance of the symmetrical set $\mathcal{K}_1(t)$ of form (6) with (5') w.r.t. the single-model system defined by A implies $\mu_{\| \, \|_{p,n}}(DAD^{-1}) \leq \alpha$, as per [8]. At the same time, there exists a matrix $A \in \mathcal{A}$ for which the equality $\mu_{\| \, \|_{p,n}}(DAD^{-1}) = \mu_{\| \, \|_{p,n}}(U)$ holds true.

The above particularizations yield results already formulated by our previous works [13], [14] that focused exclusively on symmetrical invariant sets for IMSs.

Approach to discrete-time IMSs. The analysis of non-symmetrical invariant sets can also be developed with respect to IMSs with discrete-time dynamics

$$x(t+1) = Ax(t), x(t_0) = x_0, t, t_0 \in \mathbb{Z}_+, t \geq t_0, A \in \mathcal{A}, \qquad (18)$$

where the interval matrix $\mathcal{A}$ has the same meaning as in (2). By using the non-symmetrical constant set $\mathcal{C}_p$ of form (5), we define the time-dependent sets (similar to (6), but with discrete-time description):

$$\mathcal{K}_p : \mathbb{R}_+ \rightrightarrows \mathbb{R}^n, \;\; \mathcal{K}_p(t) = \alpha^t \mathcal{C}_p, \;\; t \in \mathbb{Z}_+, \;\; 0 < \alpha \leq 1\,. \qquad (19)$$

Theorems 1 and 2 can be restated *mutatis mutandis* for discrete-time, by taking into account the following two aspects: (i) the partitioned matrix $\widetilde{A} \in \mathbb{R}^{2n \times 2n}$ (10) should be built as a non-negative matrix and the diagonal entries are constructed identically to the off-diagonal ones (see e.g. [2]); (ii) matrices $\widetilde{U}$, U_1, $U_\infty \in \mathbb{R}^{2n \times 2n}$ in (8), (12), (14) respectively, are non-negative (instead of essentially non-negative), and inequalities (9), (13), (15) are formulated in terms matrix norms (instead of matrix measures).

4. An example

Consider a second order IMS of form (1) defined by the matrices

$$\underline{A} = \begin{bmatrix} -5 & 1 \\ 4 & -6 \end{bmatrix}, \ \bar{A} = \begin{bmatrix} -3 & 2 \\ 5 & -4 \end{bmatrix}, \tag{20}$$

which was previously studied in paper [5] that proved its asymptotic stability.

We are going to explore the existence of non-symmetrical invariant sets of form (6), with $p \in \{1, 2, \infty\}$, $\widetilde{D} = diag\left\{d_1^+, d_2^+, d_1^-, d_2^-\right\} \in \mathbb{R}_+^{4\times 4}$ and the contraction rate $\alpha = -1/4$.

The use of Theorem 1 yields a unique test matrix $\tilde{U} \in \mathbb{R}^{4\times 4}$ of form (8) with $U^{11} = \begin{bmatrix} -3 & 2d_1^+/d_2^+ \\ 5d_2^+/d_1^+ & -4 \end{bmatrix}$, $U^{22} = \begin{bmatrix} -3 & 2d_1^-/d_2^- \\ 5d_2^-/d_1^- & -4 \end{bmatrix}$, $U^{12} = 0_{2\times 2}$, $U^{21} = 0_{2\times 2}$. This uniqueness is explained by the unipolarity of the intervals, as commented in Section 3. It also means that in Theorem 2 we have the matrix equality $\tilde{U} = U_1 = U_\infty$, not only the measure equality (16) used by the proof of Theorem 2.

For $p = 1$, inequality (9) means $\mu_{\|\cdot\|_{1,4}}(\widetilde{U}) \le -1/4$ and it can be satisfied if and only if $\frac{d_1^+}{d_2^+}, \frac{d_1^-}{d_2^-} \in \left[\frac{160}{88}, \frac{165}{88}\right]$. Hence, any set $\mathcal{K}_1(t)$ of form (6) defined by a matrix $\widetilde{D} \in \mathbb{R}_+^{4\times 4}$ fulfilling these conditions is invariant w.r.t. IMS (1), for instance $\widetilde{D}_1 = diag\{15, 8, 20, 11\}$.

For $p = 2$, inequality (9) means $\mu_{\|\cdot\|_{2,4}}(\widetilde{U}) \le -1/4$ and it can be satisfied if and only if $\frac{d_1^+}{d_2^+}, \frac{d_1^-}{d_2^-} \in \left[\sqrt{\frac{85-\sqrt{825}}{32}}, \sqrt{\frac{85+\sqrt{825}}{32}}\right]$. Hence, any set $\mathcal{K}_2(t)$ of form (6) defined by a matrix $\widetilde{D} \in \mathbb{R}_+^{4\times 4}$ fulfilling these conditions is invariant w.r.t. IMS (1), for instance $\widetilde{D}_2 = diag\{9, 5, 7, 5\}$.

For $p = \infty$, inequality (9) means $\mu_{\|\cdot\|_{\infty,4}}(\widetilde{U}) \le -1/4$ and it can be satisfied if and only if $\frac{d_1^+}{d_2^+}, \frac{d_1^-}{d_2^-} \in \left[\frac{32}{24}, \frac{33}{24}\right]$. Hence, any set $\mathcal{K}_\infty(t)$ of form (6) defined by a matrix $\widetilde{D} \in \mathbb{R}_+^{4\times 4}$ fulfilling these conditions is invariant w.r.t. IMS (1), for instance $\widetilde{D}_\infty = diag\{4, 3, 11, 8\}$.

There exist invariant sets $\mathcal{K}_1(t)$ and $\mathcal{K}_2(t)$ with $\alpha = -1/4$, defined by the same matrix $\widetilde{D}_{(1,2)} \in \mathbb{R}_+^{4\times 4}$. There also exist invariant sets $\mathcal{K}_2(t)$ and $\mathcal{K}_\infty(t)$ with $\alpha = -1/4$, defined by the same matrix $\widetilde{D}_{(2,\infty)} \in \mathbb{R}_+^{4\times 4}$. Nevertheless, one cannot find invariant sets $\mathcal{K}_1(t)$ and $\mathcal{K}_\infty(t)$ with $\alpha = -1/4$, defined by the same matrix $\widetilde{D}_{(1,\infty)} \in \mathbb{R}_+^{4\times 4}$. This is because for the bounds found above we have $\sqrt{\frac{85-\sqrt{825}}{32}} < \frac{32}{24} < \frac{33}{24} < \frac{160}{88} < \frac{165}{88} < \sqrt{\frac{85+\sqrt{825}}{32}}$, and Theorem 2 provides a necessary and sufficient condition for $p \in \{1, \infty\}$.

5. Conclusion

The paper develops a comprehensive framework for set-invariance analysis with respect to IMS dynamics. The main novelty of the study consists in handling several classes of invariant sets with non-symmetrical shapes, problem which was formerly addressed only for single-model systems, but not for IMSs. Thus, the current framework is able to accommodate generalizations of some older results referring to either non-symmetrical sets invariant with respect to single-model systems, or symmetrical sets invariant with respect to IMSs. The considered example uses an asymptotically stable IMS and discusses the applicability of the invariance conditions developed by our work.

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References

[1] M. Arcak and E. D. Sontag. Diagonal stability of a class of cyclic systems and its connection with the secant criterion. *Automatica*, 42(9):1531–1537, 2006.

[2] A. Benzaouia and C. Burgat. Existence of non-symmetrical Lyapunov functions for linear systems. *International Journal of Systems Science*, 20(4):597–607, 1989.

[3] D. S. Bernstein. *Matrix Mathematics: Theory, Facts, and Formulas.* Princeton University Press, 2009.

[4] F. Blanchini and S. Miani. *Set-Theoretic Methods in Control.* Springer, 2nd edition, 2015.

[5] J. Chen. Sufficient conditions on stability of interval matrices: connections and new results. *IEEE Transactions on Automatic Control*, 37(4):541–544, 1992.

[6] L. T. Gruyitch, J.-P. Richard, P. Borne, and J.-C. Gentina. *Stability domains.* Chapman Hall, CRC, 2004.

[7] E. Kaszkurewicz and A. Bhaya. *Matrix diagonal stability in systems and computation.* Birkhuser, Boston, 2000.

[8] H. Kiendl, J. Adamy, and P. Stelzner. Vector norms as Lyapunov functions for linear systems. *IEEE Transactions on Automatic Control*, 37(6):839–842, 1992.

[9] T. J. Laffey and H. Šmigoc. Tensor conditions for the existence of a common solution to the Lyapunov equation. *Linear Algebra and its Applications*, 420(2):672–685, 2007.

[10] W.-J. Mao and J. Chu. Quadratic stability and stabilization of dynamic interval systems. *IEEE Transactions on Automatic Control*, 48(6):1007–1012, 2003.

[11] M. M. Moldovan and M. S. Gowda. On common linear/quadratic Lyapunov functions for switched linear systems. In P. Pardalos, T. Rassias, and A. Khan, editors, *Nonlinear Analysis and Variational Problems*, pages 415–429. Springer, 2010.

[12] O. Pastravanu and M.-H. Matcovschi. The non-symmetrical case of invariant-set analysis with respect to linear dynamics. In *International Symposium on Signals, Circuits and Systems ISSCS 2009*, pages 65–68. IEEE, 2009.

[13] O. Pastravanu and M.-H. Matcovschi. Diagonal stability of interval matrices and applications. *Linear Algebra and its Applications*, 433(8):1646–1658, 2010.

[14] O. Pastravanu and M.-H. Matcovschi. Invariance properties of interval dynamical systems. *International Journal of Systems Science*, 42(12):1993–2007, 2011.

[15] O. Pastravanu and M. Voicu. Necessary and sufficient conditions for componentwise stability of interval matrix systems. *IEEE Transactions on Automatic Control*, 49(6):1016–1021, 2004.